

# On semigroup rings with decreasing Hilbert function.

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## Abstract

In this paper we study the Hilbert function  $H_R$  of one-dimensional semigroup rings  $R = k[[S]]$ . For some classes of semigroups, by means of the notion of *support* of the elements in  $S$ , we give conditions on the generators of  $S$  in order to have decreasing  $H_R$ . When the embedding dimension  $v$  and the multiplicity  $e$  verify  $v + 3 \leq e \leq v + 4$ , the decrease of  $H_R$  gives explicit description of the Apéry set of  $S$ . In particular for  $e = v + 3$ , we classify the semigroups with  $e = 13$  and  $H_R$  decreasing, further we show that  $H_R$  is non-decreasing if  $e < 12$ . Finally we deduce that  $H_R$  is non-decreasing for every Gorenstein semigroup ring with  $e \leq v + 4$ .

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## 0 Introduction.

Given a local noetherian ring  $(R, \mathfrak{m}, k)$  and the associated graded ring  $G = \bigoplus_{n \geq 0} (\mathfrak{m}^n / \mathfrak{m}^{n+1})$ , a classical hard topic in commutative algebra is the study of the Hilbert function  $H_R$  of  $G$ , defined as  $H_R(n) = \dim_k(\mathfrak{m}^n / \mathfrak{m}^{n+1})$ : when  $R$  is the local ring of a  $k$ -scheme  $X$  at a point  $P$ ,  $H_R$  gives important geometric information. If  $\text{depth}(G)$  is large enough, this function can be computed by means of the Hilbert function of a lower dimensional ring, but in general  $G$  is not Cohen-Macaulay, even if  $R$  has this property.

For a Cohen-Macaulay one-dimensional local ring  $R$ , it is well known that  $H_R$  is a non decreasing function when  $G$  is Cohen-Macaulay, but we can have  $\text{depth}(G) = 0$  and in this case  $H_R$  can be decreasing, i.e.  $H_R(n) < H_R(n-1)$  for some  $n$  (see, for example [8], [10], [11]). This fact cannot happen if  $R$  verifies either  $v \leq 3$ , or  $v \leq e \leq v + 2$ , where  $e$  and  $v$  denote respectively the multiplicity and the embedding dimension of  $R$  (see [6], [7], [16]). If  $R = k[[S]]$  is a semigroup ring, many authors proved that  $H_R$  is non-decreasing in several cases: •  $S$  is generated by an almost arithmetic sequence (if the sequence is arithmetic, then  $G$  is Cohen-Macaulay) [17], [13] •  $S$  belongs to particular subclasses of four-generated semigroups, which are symmetric [1], or which have Buchsbaum tangent cone [3] •  $S$  is balanced [12], [3] •  $S$  is obtained by techniques of gluing numerical semigroups [2], [9] •  $S$  satisfies certain conditions on particular subsets of  $S$  (see below) [5, Theorem 2.3, Corollary 2.4, Corollary 2.11]. If  $e \geq v + 3$ , the function  $H_R$  can be decreasing, as shown in several examples: the first one (with  $e = v + 3$ ) is in [14] (here recalled in Example 1.6). When  $G$  is not Cohen-Macaulay, a useful method to describe  $H_R$  can be found in some recent papers (see [12], [3], [5]): it is based on the study of certain subsets of  $S$ , called  $D_k$  and  $C_k$ , ( $k \in \mathbb{N}$ ).

The aim of this paper is the study of semigroup rings  $R = k[[S]]$  with  $H_R$  decreasing. To this goal we introduce and use the notion of *support* of the elements in  $S$  (1.3.4); by means of this tool we first develop a technical analysis of the subsets  $D_k$ ,  $C_k$  in Section 2. Through this machinery, under suitable assumptions on the Apéry set of  $S$ , we prove (Section 3) necessary conditions on  $S$  in order to have decreasing Hilbert function, see (3.4), (3.6).

In Section 4 we apply these results to the semigroups with  $v \in \{e - 3, e - 4\}$ .

For  $v = e - 3$  we show that the decrease of  $H_R$  is characterised by a particular structure of the sets  $C_2$ ,  $D_2$ ,  $C_3$  and that  $H_R$  does not decrease for  $e \leq 12$ , see (4.2), (4.3); in particular, for  $e = 13$ , we identify precisely the semigroups with  $H_R$  decreasing, see (4.6) and examples (4.7).

In case  $v = e - 4$  we obtain analogous informations on the structure of  $C_2$ ,  $C_3$ ,  $D_2$ ,  $D_3$ , see (4.9) and (4.10).

Such methods allow to construct various examples of semigroup rings with decreasing  $H_R$ , see, for example (3.2), (3.7) where  $e - 7 \leq v \leq e - 3$ ; in particular example (3.7.1) describes a semigroup whose Hilbert function decreases at two different levels. The examples have been performed by using the program CoCoA together with FreeMat and Excel.

As a consequence of some of the above facts, one can see that the semigroups  $S$  with  $|C_2| = 3$  and  $|C_3 \cap \text{Apéry set}| \leq 1$  cannot be symmetric. It follows that every Gorenstein ring  $k[[S]]$  with  $v \geq e - 4$  has non-decreasing Hilbert function (4.11). This result is a partial answer to the conjecture settled by M.E. Rossi [15, Problem 4.9] that a Gorenstein 1-dimensional local ring has non-decreasing Hilbert function.

# 1 Preliminaries.

We briefly recall the definition of the Hilbert function for local rings. Let  $(R, \mathfrak{m}, k)$  be a noetherian local  $d$ -dimensional ring, the *associated graded ring* of  $R$  with respect to  $\mathfrak{m}$  is  $G := \bigoplus_{n \geq 0} \mathfrak{m}^n / \mathfrak{m}^{n+1}$ ; the *Hilbert function*  $H_R : \mathbb{N} \rightarrow \mathbb{N}$  of  $R$  is defined by  $H_R(n) = \dim_k(\mathfrak{m}^n / \mathfrak{m}^{n+1})$ . This function is called *non-decreasing* if  $H_R(n-1) \leq H_R(n)$  for each  $n \in \mathbb{N}$  and *decreasing* if there exists  $\ell \in \mathbb{N}$  such that  $H_R(\ell-1) > H_R(\ell)$ , we say  $H_R$  *decreasing at level*  $\ell$ .

**1.1** Let  $R$  be a one-dimensional Cohen-Macaulay local ring and assume  $k = R/\mathfrak{m}$  infinite. Then there exists a *superficial element*  $x \in \mathfrak{m}$ , of degree 1, (i.e. such that  $x\mathfrak{m}^n = \mathfrak{m}^{n+1}$  for  $n \gg 0$ ).

It is well-known that

- $G$  is Cohen-Macaulay  $\iff$  the image  $x^*$  of  $x$  in  $G$  is a non-zero divisor.
- If  $G$  is Cohen-Macaulay, then  $H_R$  is non-decreasing.

We begin by setting the notation of the paper and by recalling some known useful facts.

**Setting 1.2** In this paper  $R$  denotes a 1-dimensional numerical semigroup ring, i.e.  $R = k[[S]]$ , where  $k$  is an infinite field and  $S = \{\sum a_i n_i, a_i, n_i \in \mathbb{N}\}$  is a *numerical semigroup* of *multiplicity*  $e$  and *embedding dimension*  $v$  minimally generated by  $\{e := n_1, n_2, \dots, n_v\}$ , with  $0 < n_1 < \dots < n_v$ ,  $\text{GCD}(n_1, \dots, n_v) = 1$ . Then  $R$  is the completion of the local ring  $k[x_1, \dots, x_v]_{(x_1, \dots, x_v)}$  of the monomial curve  $\mathcal{C}$  parametrized by  $x_i := t^{n_i}$  ( $1 \leq i \leq v$ ). The maximal ideal of  $R$  is  $\mathfrak{m} = (t^{n_1}, \dots, t^{n_v})$  and  $x_1 = t^e$  is a superficial element of degree 1. Let  $v : k((t)) \rightarrow \mathbb{Z} \cup \{\infty\}$  denote the usual valuation.

1.  $M := S \setminus \{0\} = v(\mathfrak{m})$ ,  $hM = v(\mathfrak{m}^h)$ , for each  $h \geq 1$  and the Hilbert function  $H_R$  verifies  $H_R(0) = 1$ ,  $H_R(h) = |hM \setminus (h+1)M|$ , for each  $h \geq 1$ .
2. Let  $g \in S$ , the *order* of  $g$  is defined as  $\text{ord}(g) := \max\{h \mid g \in hM\}$ .
3.  $Ap = \text{Apéry}(S) := \{s \in S \mid s - e \notin S\}$  is the *Apéry set* of  $S$  with respect to the multiplicity  $e$ ,  $|Ap| = e$  and  $e + f$  is the greatest element in  $Ap$ , where  $f := \max\{x \in \mathbb{N} \setminus S\}$  is the *Frobenius number* of  $S$ .  
Let  $d := \max\{\text{ord}(\sigma) \mid \sigma \in Ap\}$ .  
Denote by  $Ap_k := \{s \in Ap \mid \text{ord}(s) = k\}$ ,  $k \in [1, d]$  the subset of the elements of order  $k$  in  $Ap$ .
4. Let  $R' := R/t^e R$ , the Hilbert function of  $R'$  is  $H_{R'} = [1, a_1, \dots, a_d]$  with  $a_k = |Ap_k|$  for each  $k \in [1, d]$ , see, for example, [12, Lemma 1.3].
5. A semigroup  $S$  is called *symmetric* if for each  $s \in S$   $s \in Ap \iff e + f - s \in Ap$ .

By (1.2.1) and (1.1), if  $\text{ord}(s+e) = \text{ord}(s) + 1$  for each  $s \in S$ , then  $G$  is Cohen-Macaulay and  $H_R$  is non-decreasing. Therefore in order to focus on the possible decreasing Hilbert functions, it is useful to define the following subsets  $D_k, C_k \subseteq S$ , we also introduce the notion of *support* for a better understanding of these sets.

**Setting 1.3** 1.  $D_k := \{s \in S \mid \text{ord}(s) = k-1 \text{ and } \text{ord}(s+e) > k\}$ . ( $D_1 = D_k = \emptyset$ ,  $\forall k \geq r$  [12, Lemma 1.5.2]).

$D_k^t := \{s \in D_k \text{ such that } \text{ord}(s+e) = t\}$  and let  $k_0 := \min\{k \text{ such that } D_k \neq \emptyset\}$ .

2.  $C_k := \{s \in S \mid \text{ord}(s) = k \text{ and } s-e \notin (k-1)M\}$ , i.e.,  $C_k = Ap_k \cup \{\cup_h (D_h^k + e)\}$ , with  $2 \leq h \leq k-1$ .

$C_1 = \{n_2, \dots, n_v\}$ ,  $C_2 = Ap_2$ ,  $C_3 = (D_2^3 + e) \cup Ap_3$  and  $C_k = \emptyset$ ,  $\forall k \geq r+1$  [12, Lemma 1.8.1], [3].

3. A *maximal representation* of  $s \in S$  is any expression  $s = \sum_{j=1}^v a_j n_j$ ,  $a_i \in \mathbb{N}$ , with  $\sum_{j=1}^v a_j = \text{ord}(s)$

and in this case we define *support* of  $s$  as  $\text{Supp}(s) := \{n_i \in Ap_1 \mid a_i \neq 0\}$ . Recall:

$\text{Supp}(s)$  depends on the choice of a maximal representation of  $s$ . Further for  $s \in S$ , we define:

$|\text{Supp}(s)| := \max\{|\text{Supp}(\sum_i a_i n_i)|, \text{ such that } s = \sum_i a_i n_i \text{ is a maximal representation of } s\}$ .

4. For a subset  $H \subseteq S$ ,  $\text{Supp}(H) := \bigcup \{\text{Supp}(s_i), s_i \in H\}$ .

5. We call *induced by*  $s = \sum_{j=1}^v a_j n_j$  (maximal representation) an element

$$s' = \sum_{j=0}^v b_j n_j, \text{ with } 0 \leq b_j \leq a_j. \quad \text{Recall: } \text{ord}(s') = \sum b_j \text{ by [12, Lemma 1.11].}$$

The following two propositions are crucial in the sequel.

**Proposition 1.4** *Let  $S$  be as in Setting 1.2 and let  $s \in C_k$ ,  $k \geq 2$ . Then*

1. For each  $s = \sum_{j \geq 1} a_j n_j \in C_k$  (maximal representation, with  $\sum_{j \geq 1} a_j = k$ ), we have:
  - (a) Every element  $s' = \sum_{j \geq 1} b_j n_j$  induced by  $s$ , with  $\sum_{j \geq 1} b_j = h$ , belongs to  $C_h$ .
  - (b)  $\text{Supp}(C_k) \subseteq \text{Supp}(C_{k-1}) \subseteq \dots \subseteq \text{Supp}(C_2)$ .
2. If  $s = g + e$  with  $g \in D_h$ , ( $h \leq k-1$ ), any maximal representation  $s = \sum_{j \geq 1} a_j n_j$  has  $a_1 = 0$ .

Proof. (1.a). See [12, Lemma 1.11].

(2). If  $a_1 > 0$ , then  $g = (a_1 - 1)e + \sum_{j \geq 2} a_j n_j$  has  $\text{ord} = k - 1$ , by (1.a), contradiction.  $\diamond$

**Proposition 1.5** *Let  $S$  be as in Setting 1.2, let  $k_0$  be as in (1.3.1) and let  $k \geq 2$ .*

1.  $H_R(k) - H_R(k-1) = |C_k| - |D_k|$  [12, Proposition 1.9.3], [3, Remark 4.1]
2.  $G$  is Cohen Macaulay  $\iff D_k = \emptyset$  for each  $k \geq 2$ . [12, Theorem 1.6].
3. If  $|D_k| \leq k+1$  for every  $k \geq 2$ , then  $H_R$  is non-decreasing [5, Theorem 3.3, Corollary 3.4].
4. If  $|D_k| \geq k+1$ , then  $|C_h| \geq h+1$  for all  $h \in [2, k]$  [5, proof of Proposition 3.9].
5. In particular (recall:  $k_0 = \min\{k \mid D_k \neq \emptyset\}$ ) [5, Corollary 3.11]:
  - (a) If  $H_R$  is decreasing at level  $k$ , then  $|D_k| \geq \max\{1 + |C_k|, k+2\}$ ,
  - (b) If  $H_R$  is decreasing, then  $|Ap_{k_0}| \geq k_0 + 1$ ,
  - (c) If  $H_R$  is decreasing, then  $|Ap_2| \geq 3$ .

We show a semigroup  $S$  with  $H_R$  decreasing, which is the *first example* in the sense that  $e = v + 3 = 13$ ,  $|Ap_2| = 3$  are minimal in order to have decreasing Hilbert function, see (4.2), (4.3).

**Example 1.6** [14, Section 2] *Let  $S = \langle 13, 19, 24, 44, 49, 54, 55, 59, 60, 66 \rangle$ , with  $e = 13$ ,  $v = 10$ ,  $Ap = \{ 0 \ 19, \ 24, \ 38, \ 43, \ 44, \ 48, \ 49, \ 54, \ 55, \ 59, \ 60, \ 66 \}$ .*

$M \setminus 2M$	13	19	24				<b>44</b>		<b>49</b>	<b>54</b>	55	<b>59</b>	60	66	<i>dim</i>	=	10
$2M \setminus 3M$	26	32	37	<b>38</b>	<b>43</b>		<b>48</b>				68		73	79	<i>dim</i>	=	9
$3M \setminus 4M$	39	45	50	51	56	57	61	62	67			72		92	<i>dim</i>	=	11
$4M \setminus 5M$	52	58	63	64	69	70	74	75	80	81	85	86			<i>dim</i>	=	12
$5M \setminus 6M$	65	71	76	77	82	83	87	88	93	94	98	99	105		<i>dim</i>	=	13

Then  $H_R = [1, 10, 9, 11, 12, 13 \rightarrow]$ ,  $H_{R'} = [1, 9, 3]$ . Further  $Supp(D_2 + e) = Supp C_2 = \{19, 24\}$ :

$\mathbf{D_2}$	$= \{44, 49, 54, 59\}$	$\mathbf{C_2} = \{38, 43, 48\} = \{19 \cdot 2, 19 + 24, 24 \cdot 2\} = Ap_2$
$D_2 + e$	$= \{57, 62, 67, 72\}$	$57 = 3 \cdot 19, \quad 62 = 2 \cdot 19 + 24, \quad 67 = 19 + 2 \cdot 24, \quad 72 = 3 \cdot 24$
$D_3$	$= \{68, 73\}$	$C_3 = \{57, 62, 67, 72\} = D_2 + e$
$D_3 + e$	$= \{81, 86\}$	$81 = 3 \cdot 19 + 24, \quad 86 = 2 \cdot 19 + 2 \cdot 24$
$D_4$	$= \{92\}$	$C_4 = \{81, 86\} = D_3 + e$
$D_4 + e$	$= \{105\}$	$105 = 3 \cdot 19 + 2 \cdot 24$
$D_5$	$= \{\emptyset\}$	$C_5 = \{105\} = D_4 + e.$

## 2 Technical analysis of $C_k$ and $D_k$ via supports and Apéry subsets.

**Lemma 2.1** Let  $x = \sum_{i=2}^v a_i n_i \in C_k$ , with  $a_i \geq 1$ , for each  $i$ ,  $\sum_{i=2}^v a_i = \text{ord}(x) = k$  and  $|\text{Supp}(x)| = q$ .

Let  $2 \leq h < k$ , then 
$$\begin{cases} (a) & |C_h| \geq hp + 1 \geq q, & \text{if } q \geq h + 1, p \geq 1 \\ (b) & |C_h| \geq q, & \text{if } q \leq h. \end{cases}$$

Proof. First recall that, by (1.4.1a), every element of order  $h$  induced by the given maximal representation of  $x$ , belongs to  $C_h$ . We denote for simplicity  $\text{Supp}(x) = \{m_1 < m_2 < \dots < m_q\}$ , distinct minimal generators, with  $m_i \neq e$  by (1.4.2).

(a). If  $q \geq h + 1$ .

Then we can construct the  $(h + 1) + h(q - h - 1)$  distinct induced elements in  $C_h$ :

$$\left\{ \sigma_{\eta, i} = (\sum_{j=\eta}^{\eta+h} m_j) - m_i, \quad \eta = 1, \dots, q - h, \begin{cases} i = 1, \dots, h + 1, & \text{for } \eta = 1, \\ i = \eta, \dots, h + \eta - 1, & \text{for } 2 \leq \eta \leq q - h, \end{cases} \right\}.$$

(b). If  $1 \leq q \leq h$ , there exists  $(a'_1, \dots, a'_q)$  such that  $a_i \geq a'_i \geq 1, \forall i = 1, \dots, q, \sum_i a'_i = h + 1$ . Then  $C_h$  contains the  $q$  distinct elements  $\sigma_j = \sum_1^q a'_i m_i - m_j, j = 1, \dots, q$ .  $\diamond$

**Proposition 2.2** Let  $k_0 = \min\{k \in \mathbb{N} \mid D_k \neq \emptyset\}$ ,  $d = \max\{\text{ord}(\sigma), \sigma \in Ap\}$ .

1. Let  $g \in D_k, g + e = \sum_j a_j n_j$ , with  $\sum_j a_j = k + p, p \geq 1$  (maximal representation):

- (a) Let  $y \in C_h, h < k + p$ , be induced by  $g + e$ ; if  $h \leq \max\{p + 1, k_0\}$ , then  $y$  belongs to  $Ap$ .
- (b)  $p \leq d - 1$ .
- (c)  $|\text{Supp}(g + e)| \leq |Ap_{p+1}|$ .

2. If  $Ap_3 = \emptyset$ , then  $D_k + e = C_{k+1}$  for each  $k \geq 2$ .

Proof. 1.(a) Let  $h \leq k_0$  and let  $z \in C_h \setminus Ap$ ; then  $z - e \in S$ , with  $\text{ord}(z - e) \leq h - 2$ , hence  $z - e \in D_r, r < k_0$ , impossible. Further, if  $y = \sum b_j n_j$  and  $y \notin Ap$ , with  $h = \sum b_j \leq p + 1$ , then  $y = e + \sigma$ , with  $0 < \sigma \in S$ . Then:  $g = \sigma + \sum (a_j - b_j) n_j \implies \text{ord}(g) \geq 1 + k + p - (p + 1) = k$ , contradiction.

1.(b) Let  $g + e = \sum_j a_j n_j$  (maximal representation), if  $\sum_j a_j \geq k + d$ , then  $d + 1 \leq \sum_j a_j - k + 1$  and so every induced element  $\in C_{d+1}$  belongs to  $Ap$  by (1), impossible, since  $(d + 1)M \subseteq M + e$ ; hence  $\text{ord}(g + e) = k + 1$ .

1.(c) Let  $|\text{Supp}(g + e)| = q$  and let  $h := p + 1 < p + k$ , by (2.1) there are at least  $q$  elements in  $C_h$ . These elements are in  $Ap$ , by 1.(a). Hence  $q \leq |Ap_{p+1}|$ .

2. If there exists  $g \in D_k$  such that  $\text{ord}(g + e) \geq k + 2$ , then  $p \geq 2$ ; by 1.(a) we would have  $y \in Ap_3$  for every  $y \in C_3$  induced by  $g + e$ .  $\diamond$

**Remark 2.3** 1. (2.2.1 a) cannot be improved, for example in (1.6) if  $s = 92$ :  $\text{ord}(s) = 3, 92 + e = 105 = 3 \cdot 19 + 2 \cdot 24 \implies 92 \in D_4, p = 1$  and  $2 \cdot 19 + 24 = 62 = 49 + e \in C_3 \setminus Ap_3$  (here  $3 = p + 2$ ).

2. Let  $x_1, x_2 \in D_k, x_1 \neq x_2$  such that  $\text{Supp}(x_1 + e) = \text{Supp}(x_2 + e), x_1 + e = \sum \alpha_i n_i, x_2 + e = \sum \beta_i n_i$ . Then there exist  $i, j$  such that  $\alpha_i > \beta_i, \alpha_j < \beta_j$ .

Proof. If  $x_1 \neq x_2$  and  $\alpha_i \geq \beta_i$  for each  $i$ , then  $x_1 + e = x_2 + e + \sigma, \text{ord}(\sigma) \geq 1$ , hence  $\text{ord}(x_1) > \text{ord}(x_2)$ , impossible.

3. Let  $x \in D_k, \text{ord}(x + e) \geq k + 2$ . If  $y \in D_k$  and  $\text{ord}(y + e) = h \leq k + 1$ . Then  $y + e$  cannot be induced by  $x + e$ .

Proof. If  $x + e = y + e + s$ , with  $\text{ord}(s) \geq 1$  then  $x = y + s, \text{ord}(x) > \text{ord}(y)$ , impossible, since by assumption  $\text{ord}(x) = \text{ord}(y) = k - 1$ .

Given the sets  $C_h, C_k$ , with  $h < k$ , we estimate lower bounds for the cardinality of  $C_h$ , by enumerating the elements induced by  $C_k$ . We first consider the elements induced by the subset  $\{x \in C_k \text{ such that } |\text{Supp}(x)| \leq 2\}$ .

**Lemma 2.4** For  $x_r \in C_k, (k \geq 3)$ , let  $x_r = a_r n_i + b_r n_j, a_r + b_r = k$  and let  $r \in [2, k - 1]$ .

1. Let  $x_1 \in C_k$ ; the number of distinct elements of  $C_h$  induced by  $x_1$  is

$$\beta_1 = 1 + \min\{a_1, b_1, h, k - h\} = \begin{cases} 1 + \min\{b_1, h\}, & \text{if } a_1 \geq h \\ 1 + a_1 & \text{if } a_1 < h \leq b_1 \\ 1 + k - h & \text{if } a_1, b_1 < h \end{cases}$$

with  $\beta_1 \geq 1$  and  $\beta_1 = 1 \iff a_1 b_1 = 0$ .

2. Let  $x_1, x_2$  be distinct elements in  $C_k$  such that  $\text{Supp}(x_1) \cup \text{Supp}(x_2) = \{n_i, n_j\}$ .

$$\text{We can assume that } \begin{cases} x_1 = a_1 n_i + b_1 n_j, \\ x_2 = a_2 n_i + b_2 n_j, \\ a_i + b_i = k, \quad 0 \leq a_1 < a_2, \implies 0 \leq b_2 < b_1. \end{cases}$$

The number of distinct elements of  $C_h$  induced by  $\{x_1, x_2\}$  is

$$\beta_2 = \begin{cases} (a) & 1 + \min\{b_1, h\} \geq 2 \quad (= 2 \iff b_1 = 1, b_2 = 0) & \text{if } h \leq a_1 < a_2 \\ (b) & 2 + \min\{a_1, k - h\} + \min\{h - a_1 - 1, b_2\} \geq 3 & \text{if } 0 \leq a_1 < h \leq a_2 \\ (c) & 2 + \min\{a_1, k - h\} + \min\{a_2 - a_1 - 1, k - h\} \geq 2 & \text{if } 0 \leq a_1 < a_2 < h \\ & \beta_2 = 2 \iff a_1 = 0, a_2 = 1 \end{cases}$$

$$3. \text{ If } C_k \supseteq \{x_1, x_2, x_3\}, \text{ let } \begin{cases} x_1 = a_1 n_i + b_1 n_j \\ x_2 = a_2 n_i + b_2 n_j, \\ x_3 = a_3 n_i + b_3 n_j, \\ a_i + b_i = k, \quad 0 \leq a_1 < a_2 < a_3, \quad b_1 > b_2 > b_3 \geq 0 \end{cases},$$

The number of distinct elements of  $C_h$  induced by  $\{x_1, x_2, x_3\}$  is

$$\beta_3 = \begin{cases} (a) & 1 + \min\{b_1, h\} \geq 3 & \text{if } h \leq a_1 < a_2 < a_3 \\ (b) & 2 + \min\{a_1, k - h\} + \min\{h - a_1 - 1, b_2\} \geq 3 & \text{if } a_1 < h \leq a_2 < a_3 \\ & \text{further } \beta_3 \geq 4, \text{ except the cases :} \\ & \quad (i) \quad h = 2, a_1 \in \{0, 1\} \\ & \quad (ii) \quad k = h + 1, a_1 = h - 1, b_1 = 2 \\ & \quad (iii) \quad a_1 = 0, b_2 = 1 \\ (c) & 3 + \min\{a_1, k - h\} + \min\{a_2 - a_1 - 1, k - h\} + \min\{h - a_2 - 1, b_3\} & \text{if } a_1 < a_2 < h \leq a_3 \\ & \text{further } \beta_3 \geq 4, \text{ except the cases :} \\ & \quad (c_1) \quad a_1 = a_2 - 1 = \min\{h - 2, b_3\} = 0 \\ (d) & 3 + \min\{a_1, k - h\} + \sum_{i=1,2} \min\{a_{i+1} - a_i - 1, k - h\} & \text{if } a_1 < a_2 < a_3 < h \\ & \text{further } \beta_3 = 3 \iff a_1 = 0, a_2 = 1, a_3 = 2 \end{cases}$$

Proof. 1. Put  $a_1 = a, b_1 = b$  for simplicity. If  $a \geq h$ , the induced distinct elements in  $C_h$  are:

$\{hn_i, (h-1)n_i + n_j, \dots, hn_j\}$ , if  $h \leq b$ ,

$\{hn_i, (h-1)n_i + n_j, \dots, bn_j\}$ , if  $h > b$ .

Then  $\beta_1 = 1 + \min\{b, h\} = 1 + \min\{a, b, h, a + b - h\}$ .

If  $a < h \leq b$ , then  $\{hn_i, (h-1)n_i + n_j, \dots, (h-a)n_i + an_j\}$  are induced distinct elements of  $C_h$ . Hence  $\beta_1 = a + 1$  where  $a = \min\{a, b, h, a + b - h\}$  since  $a \leq a = b - h < b$ .

If  $a, b < h$ , then the induced distinct elements are:  $an_i + (h-a)n_j, (a-1)n_i + (h-a+1)n_j, \dots, (h-b)n_i + bn_j$ , then  $\beta_1 = a + b - h + 1 = k - h + 1$ , further  $\beta_1 = 1 + \min\{a, b, h, a + b - h\} = a + b - h (= k - h)$ , since  $a + b - h < a < h, a + b - h < b$ .

If  $a, b \geq h$ , then  $\beta_1 = 1 + h$  and the induced distinct elements are  $\{hn_i, (h-1)n_i + n_j, \dots, hn_j\}$ . In this case  $|C_h| = h + 1$  is maximal. Then, if  $a \geq h, \beta_1 = 1 + \min\{b_1, h\}$ .

2(a). The induced distinct elements  $\in C_h$  are  $\{(h-i)n_i + in_j, 0 \leq i \leq \min\{b_1, h\}\}$ .

2(b). We have  $1 + \min\{a_1, k - h\}$  elements  $\in C_h$  induced by  $x_1$ , by (1):

$$(a_1 - i)n_i + (h - a_1 + i)n_j, \quad 0 \leq i \leq \min\{a_1, k - h\}$$

Moreover from  $x_2$  we can extract the following distinct additional elements

$(h-p)n_i + pn_j$ , ( $p \geq 0$ ,  $h-p \geq a_1 + 1$ ,  $p \leq b_2$ ) hence with  $0 \leq p \leq \min\{h-a_1-1, b_2\}$ .

2(c). First,  $x_1$  induces the same elements of  $C_h$  considered in (2.b); moreover from  $x_2$  one gets other  $M$  distinct elements

$$(a_2-p)n_i + (h-a_2+p)n_j, \text{ with } p \geq 0, a_2-p \geq a_1+1, h-a_2+p \leq b_2$$

(hence  $0 \leq p \leq \min\{a_2-a_1-1, k-h\}$ ), where

either  $M = (a_2-a_1)$ , if  $h-a_1-1 \leq b_2$  (elements  $(a_2-p)n_i + (h-a_2+p)n_j$ , with  $0 \leq p \leq a_2-a_1-1$ );  
or  $M = k-h+1$ , if  $h-a_1-1 > b_2$  (elements  $a_2n_i + (h-a_2)n_j, \dots, (h-b_2)n_i + b_2n_j$ ).

3. It comes directly by using the same ideas of the proof of statement (2).  $\diamond$

This lemma allows to prove the more general

**Proposition 2.5** Assume  $k \geq 3$ ,  $2 \leq h \leq k-1$  and  $C_k \supseteq \{x_1, x_2, \dots, x_p\}$ ,  $p \leq k+1$ , with

$$\begin{cases} x_i = a_i n_i + b_i n_j, & 1 \leq i \leq p, \quad a_i + b_i = k \\ 0 \leq a_1 < a_2 < \dots < a_p \leq k & (\implies k \geq b_1 > b_2 > \dots > b_p \geq 0) \end{cases}$$

and let  $\beta_p$  be the number of distinct elements of  $C_h$  induced by  $\{x_i, i = 1, \dots, p\}$ . Then  $\beta_p \geq \min\{h+1, p\}$ , precisely:

$$\begin{cases} 1 + \min\{b_1, h\} & \text{if } h \leq a_1 < \dots < a_p \\ i + 1 + \min\{a_1, k-h\} + \dots + \min\{a_i - a_{i-1} - 1, k-h\} + \min\{h-a_i-1, b_{i+1}\} & \text{if } < a_i < h \leq a_{i+1} < \\ & (i \leq p-1) \\ p + \min\{a_1, k-h\} + \dots + \min\{a_p - a_{p-1} - 1, k-h\} & \text{if } < a_p < h \end{cases}$$

**Lemma 2.6** Let  $x_1, x_2$  be distinct elements in  $D_k$  such that  $\text{Supp}(x_1+e) = \{n_i, n_j\}$ ,  $\text{Supp}(x_2+e) = \{n_t, n_u\}$ ,  $\{n_i, n_j\} \cap \{n_t, n_u\} = \emptyset$ . Let

$$\begin{cases} y_1 = x_1 + e = an_i + bn_j \\ y_2 = x_2 + e = cn_t + dn_u \\ abcd \neq 0, a+b = k+r_1, c+d = k+r_2 \end{cases}.$$

Let  $r := \min\{r_1, r_2\}$ , and let  $2 \leq h \leq k+r$ . Consider the induced elements  $z_1 = pn_i + (h-p)n_j$ ,  $z_2 = qn_t + (h-q)n_u \in C_h$ .

Then  $z_1 \neq z_2$  for every  $p, q, h$  and  $|C_h| \geq \beta_{ab} + \beta_{cd}$  where  $\beta_{ab}, \beta_{cd}$  are defined in (2.4.1) (called  $\beta_1$ ). Consequently

$$|C_h| \geq 4, \text{ if } h < k+r, \quad |C_h| \geq 3, \text{ if } h = k+r, \quad r_1 \neq r_2, \quad |C_h| \geq 2, \text{ if } h = k+r, \quad r_1 = r_2.$$

Proof. Assume  $z_1 = z_2$ , then by substituting we get

$$\begin{cases} y_1 = (a-p)n_i + (b+p-h)n_j + z_1 = (a-p)n_i + (b+p-h)n_j + qn_t + (h-q)n_u \\ y_2 = (c-q)n_t + (d+q-h)n_u + z_1 = pn_i + (h-p)n_j + (c-q)n_t + (d+q-h)n_u \end{cases}.$$

First note that by the assumption, we have  $z_1 \neq z_2$  if:  $p = q = 0$ , or  $p = q - h = 0$ , or  $q = h - p = 0$ , or  $h = p = q$ ; moreover if three coefficients are  $\neq 0$ , then  $|\text{Supp}(y_i)| = 3$ , against the assumption. This argument allows to complete the proof in the remaining cases which are the following:

$$\begin{array}{cccc|cccc} (a-p)n_i & (b+p-h)n_j & qn_t & (h-q)n_u & pn_i & (h-p)n_j & (c-q)n_t & (d+q-h)n_u \\ \neq 0 & 0 & 0 & \neq 0 & \neq 0 & b & c & d-h \\ \neq 0 & 0 & \neq 0 & 0 & \neq 0 & b & c-h & d \\ 0 & \neq 0 & 0 & \neq 0 & \neq 0 & \neq 0 & c & d-h \\ 0 & \neq 0 & \neq 0 & 0 & \neq 0 & \neq 0 & c-h & d \end{array} \quad \diamond$$

**Lemma 2.7** Let  $k \geq 3$  and let  $x_1, x_2$  be distinct elements in  $D_k$  such that  $|\text{Supp}(x_1+e)| = |\text{Supp}(x_2+e)| = 2$ ,

$$\begin{cases} x_1 + e = an_i + bn_j \\ x_2 + e = cn_t + dn_j \end{cases}, \quad a, b, c, d > 0 \text{ with } n_i, n_j, n_t \text{ distinct elements in } Ap_1.$$

For  $h < k$ , consider the induced elements in  $C_h : \{ z = pn_i + (h-p)n_j, \quad z' = qn_t + (h-q)n_j \}$ . Then

$$1. \quad z \neq z' \text{ in the following cases } \begin{cases} (i) & pq = 0 \quad \text{and} \quad p+q > 0 \\ (ii) & a < h \quad \text{and} \quad q \geq \max\{1, h-d\} \\ & \text{or} \\ & c < h \quad \text{and} \quad p \geq \max\{1, h-b\} \end{cases}$$



2. In the cases  $\begin{cases} x_1 + e = n_i + bn_j \\ x_2 + e = n_t + dn_j \end{cases}$  we have  $|C_h| \geq 3$ , in the remaining cases  $|C_h| \geq 4$ .

Proof. 1. (i) is immediate by the assumptions.

(ii). It is enough to consider  $p > 0$  by (i); if  $z = z'$ , then

$x_2 + e = pn_i + (d + q - p)n_j + (c - q)n_t$ , where  $d + q - p \geq h - p \geq 1$ , since  $p \leq a < h$ . Then:

$$\begin{cases} |Supp(x_2 + e)| \geq 3 & \text{if } c - q > 0 \\ Supp(x_2 + e) = Supp(x_1 + e) & \text{if } c - q = 0 \end{cases}, \quad \text{contradiction in any case.}$$

2. We can assume  $0 < a \leq c$ . The following  $z_i$  belong to  $C_h$  and are distinct.

$$\begin{aligned} (i) \quad a = c = 1: \quad & z_1 = n_i + (h - 1)n_j \\ & z_2 = h n_j \\ & z_3 = n_t + (h - 1)n_j \\ (ii) \quad 1 \leq a, c < h: \quad & z_1 = an_i + (h - a)n_j \\ & (ac \geq 2) \quad z_2 = (a - 1)n_i + (h - a + 1)n_j \\ & z_3 = cn_t + (h - c)n_j \\ & z_4 = (c - 1)n_t + (h - c + 1)n_j \\ (iii) \quad a < h = c: \quad & z_1 = an_i + (h - a)n_j \\ & z_2 = (a - 1)n_i + (h - a + 1)n_j \\ & z_3 = c n_t \\ & z_4 = (c - 1)n_t + n_j \\ (iv) \quad a < h < c: \quad & z_1 = an_i + (h - a)n_j \\ & z_2 = (a - 1)n_i + (h - a + 1)n_j \\ & z_3 = h n_t \\ & z_4 = (h - 1)n_t + n_j \\ (v) \quad h \leq a \leq c: \quad & z_1 = hn_i \\ & z_2 = (h - 1)n_i + n_j \\ & z_3 = h n_t \\ & z_4 = (h - 1)n_t + n_j \end{aligned}$$

The non trivial subcases of (i),  $\dots$ , (iv) come directly from part 1.

Case (v).  $z_1 = z_4 \implies$  either  $Supp(x_1 + e) = Supp(x_2 + e)$  (if  $a = h$ ), or  $|Supp(x_1 + e)| = 3$ , (if  $a > h$ ), against the assumptions.

$z_2 = z_3 \implies$  either  $Supp(x_2 + e) = Supp(x_1 + e)$  (if  $c = h$ ), or  $|Supp(x_2 + e)| = 3$ , (if  $c > h$ ), against the assumptions.  $\diamond$

Next proposition shows that for  $k = 2$ , statement (1.5.4) holds also at step  $k + 1$ .

**Proposition 2.8** Assume  $H_R$  decreasing at level  $h$ . Then

1. There exist  $x_1, x_2 \in D_h$  such that  $|Supp(x_r + e)| \geq 2$ , for  $r = 1, 2$ .
2.  $|C_3| \geq 4$ .

Proof. 1. Assume  $H_R$  decreasing at level  $h$ : we know that  $m = |C_h| \leq |D_h| - 1$ . If  $|Supp(x_r + e)| = 1 \forall x_r \in D_h$ , then  $x_r + e = \alpha_r m_r$ ,  $\alpha_r \geq h + 1$ ,  $m_r \in Ap_1$ , with  $m_r \neq m_s$ , if  $r \neq s$ . Hence the elements  $\{hm_r\}$  would be distinct elements  $\in C_h$  and  $|C_h| > m$ .

Hence let  $x_1 \in D_h$ ,  $|Supp(x_1 + e)| \geq 2$ , with a maximal representation  $x_1 + e = \sum \beta_i m_i$ , with  $\beta_i \geq 1$ ,  $m_i \in Ap_1$  distinct elements. One induced element  $\in C_h$  is  $c_1 = m_i + m_j + s$  for some  $s$  of order  $h - 2$ . If each  $x_i \in D_h$ ,  $i > 1$  has maximal representation of the type  $x_i + e = \alpha_i n_i$ , then  $hm_j, \dots, hm_{m+1} \in C_h$  are distinct elements; further  $hm_i \neq c_1$  for each  $i$  (otherwise  $x_i + e = c_1 + (\alpha_i - h)m_i$  and so  $|Supp(x_i + e)| \geq 2$ ). Then  $|C_h| \geq m + 1$ , contradiction.

2. By (1.5.3 and 4) we know that if  $h > 2$  then  $|C_j| \geq j + 1$  for all  $2 \leq j \leq h$ . It remains to prove the statement for  $h = 2$ , and  $|D_2| \geq 4$  by [5, prop.2.4].

Since  $C_3 = Ap_3 \cup (D_2^3 + e)$ , the fact is true if  $ord(x + e) = 3 \forall x \in D_2$ . Hence we assume there exists  $x \in D_2$  such that  $ord(x + e) \geq 4$ .

In this proof, for simplicity, we denote the elements of  $Supp(x)$  with  $m_1, m_2, m_3, \dots$  and assume  $m_1 < m_2 < m_3$ .

If  $|Supp(x + e)| \geq 4$ :  $x + e = m_1 + m_2 + m_3 + m_4 + s$ , with  $m_1 < m_2 < m_3 < m_4$ , then  $|C_3| \geq 4$  by (2.1.a).

If  $|Supp(x + e)| = 3$ ,  $x + e = \beta_1 m_1 + \beta_2 m_2 + \beta_3 m_3$  with  $\beta_i \geq 1$ ,  $\sum \beta_i \geq 5$ , we find 4 induced elements in  $C_3$  as follows:

when  $\beta_1 \geq 2, \beta_2 \geq 2$ :  $\{m_1 + m_2 + m_3, 2m_1 + m_2, 2m_1 + m_3, 2m_2 + m_3\}$ ;

when  $\beta_1 \geq 2, \beta_3 \geq 2$ :  $\{m_1 + m_2 + m_3, 2m_1 + m_2, 2m_1 + m_3, m_2 + 2m_3\}$ ;  
 when  $\beta_1, \beta_2 \geq 2, \beta_3 \geq 2$ :  $\{m_1 + m_2 + m_3, 2m_2 + m_3, m_1 + 2m_2, m_2 + 2m_3\}$ ;  
 when  $\beta_1 \geq 3$ :  $\{m_1 + m_2 + m_3, 2m_1 + m_2, 2m_1 + m_3, 3m_1\}$ .

If  $|Supp(x+e)| = 3$ ,  $x+e = \beta_1 m_1 + \beta_2 m_2 + \beta_3 m_3$  with  $\beta_i \geq 1$ ,  $\sum \beta_i = 4$ , then  $x+e$  induces in  $C_3$  three distinct elements  $c_1, c_2, c_3$  (see the following table). To get the fourth element, we take  $y \in D_2, y \neq x, |Supp(y+e)| \geq 2$  (by 1). If  $Supp(x+e) \neq Supp(y+e)$  and each element induced by  $y+e$  in  $C_3$  is also induced by  $x+e$ , then we can always reduce to the case  $Supp(x+e) \supseteq Supp(y+e)$ , by suitable substitutions (in one or more steps). Hence we can assume  $Supp(x+e) \supseteq Supp(y+e)$ ; we study in the next table one of the possible subcases, the remaining are similar. Let

$$\begin{cases} x+e = m_1 + 2m_2 + m_3 \\ y+e = \beta'_1 m_1 + \beta'_2 m_2 + \beta'_3 m_3, \quad \sum \beta'_i \geq 3 \end{cases}$$

If  $\sum \beta'_i = 3$ , then  $y+e \neq z$  for each  $z \in C_3$  induced by  $x+e$ , otherwise  $ord(x) > ord(y)$ . Hence assume  $\sum \beta'_i > 3$ ;

$$\text{then } C_3 \supseteq \begin{cases} c_1 = m_1 + 2m_2 \\ c_2 = m_1 + m_2 + m_3 \\ c_3 = 2m_2 + m_3 \\ \text{and, according to the values of the } \beta'_i, \\ c_4 = 2m_1 + m_2 & \text{if } \beta'_1 = 2, \beta'_2 \geq 1 \\ c_5 = m_2 + 2m_3 & \text{if } \beta'_1 \leq 1, 1 \leq \beta'_2 \leq 2, \beta'_3 = 2 \\ c_6 = 3m_3 & \text{if } \beta'_3 \geq 3 \\ c_7 = 3m_1 & \text{if } \beta'_1 \geq 3 \\ c_8 = 3m_2, \text{ or } 2m_1 + m_2, \text{ or } m_2 + 2m_3 & \text{if } \beta'_2 \geq 3 \\ c_9 = 2m_1 + m_3 & \text{if } \beta'_1 = 2, \beta'_2 = 0, \beta'_3 \geq 1 \end{cases}$$

Clearly the elements  $c_i, i = 4, \dots, 7$  are distinct from  $c_1, c_2, c_3$ . In case  $\beta'_2 \geq 3$  we have either  $y+e = 3m_2 + m_i$ , ( $i = 1$  or  $3$ ) or  $y+e = 3m_2$ . In the first case if  $3m_2 = m_1 + m_2 + m_3$ , then  $2m_2 = m_1 + m_3 \implies y+e = 2m_1 + m_2 + m_3$ , or  $y+e = m_1 + m_2 + 2m_3$  and we can add  $c_4$  resp.  $c_5$ .

In case  $y+e = 3m_2$ , we get  $3m_2 \notin \{c_1, c_2, c_3\}$ , otherwise  $2m_2 = m_1 + m_3 \implies y+e = m_1 + m_2 + m_3$ , i.e.,  $x = y + m_2$ , impossible since  $ord(x) = ord(y)$ .

If  $\beta'_1 = 2, \beta'_2 = 0, \beta'_3 \geq 1$ , then  $c_9 \notin \{c_1, c_2, c_3\}$  otherwise  $2m_1 + m_3 = m_1 + 2m_2 \implies m_1 + m_3 = 2m_2 \implies x - y = (2 - \alpha)m_3$ , impossible in any case.

Finally we assume:  $|Supp(x+e)| = |Supp(y+e)| = 2$ ,  $ord(x+e) \geq 4$ . By lemmas (2.6) and (2.7), it remains to analyse the cases

$$(I) \begin{cases} x+e = m_1 + bm_2 \\ y+e = m_3 + dm_2 \\ b \geq 3, d \geq 2 \end{cases} \quad (II) \begin{cases} x_1+e = am_1 + bm_2 \\ x_2+e = cm_1 + dm_2 \\ a+b \geq 4, c+d \geq 3 \end{cases} \quad (III) \begin{cases} x+e = am_1 + bm_2 & a+b=3 \\ y+e = cm_3 + dm_4 & c+d \geq 4 \end{cases}$$

(I). The induced distinct elements are  $\{c_1 = m_1 + 2m_2, c_2 = 3m_2, c_3 = 2m_2 + m_3\}$ .

By assumption, there exist two other elements  $z_1, z_2 \in D_2$ ; by the above tools and by (2.3), we can restrict to the cases

$$b \geq 3, d \geq 2 : \begin{cases} (i) & z_1+e = \alpha m_1 + \beta m_2 & \alpha \neq 1 & \alpha + \beta \geq 3 \\ (ii) & z_1+e = \alpha m_3 + \beta m_2 & \alpha \neq 1 & \alpha + \beta \geq 3 \\ (iii) & z_1+e = \alpha m_1 + \beta m_3 & \alpha + \beta \geq 3 \end{cases}$$

In the first two cases, if  $\alpha > 0$ , then  $\alpha \geq 2$  (by 2.3.2), therefore if  $\beta > 0$  we obtain  $|C_3| \geq 4$  by applying (2.7.2) to the pair of elements  $\{z_1+e, y+e\}$  (resp  $\{x+e, z_1+e\}$ ). In case (iii), again, from (2.7.2), by substituting in  $\{x+e, z_1+e\}$   $m'_1 = m_2, m'_2 = m_1, m'_3 = m_3$ , we deduce  $|C_3| \geq 4$ , if  $\alpha\beta > 0$ . The remaining possibilities are

$$\begin{aligned} (i') & \quad z_1+e = \beta m_2 & \beta \geq 3 \\ (ii') & \quad z_1+e = \alpha m_1 & \alpha \geq 3 \\ (iii') & \quad z_1+e = \alpha m_3 & \alpha \geq 4 \end{aligned}$$

In case (i') we can assume the element  $z_2 \in D_2$ , verifies  $z_2+e = \alpha m_1$ ,  $\alpha \geq 3$  (or  $z_2+e = \alpha m_3$ ).

If  $3m_1 = 2m_2 + m_3$ , then, either  $|Supp(z_2+e)| = 3$  and we are done, or  $z_2+e = 2m_2 + m_3$ , impossible since  $y+e = m_3 + dm_2$ .

Similarly we can solve case (ii').

In case (iii'), if  $3m_3 = m_1 + 2m_2$ , then  $z_1+e = m_1 + 2m_2 + (\alpha-3)m_3$ . The case  $\alpha \geq 5$  has already been proved above. If  $\alpha = 4$ , we consider  $m_1 + m_2 + m_3 \in C_3$  which, in this situation, is distinct from  $c_1, c_2, c_3$ .



(II). If we have 4 elements with  $\text{support} \subseteq \{m_1, m_2\}$ , then we are done by (2.5). The other cases can be studied among the ones of shape (I) or (III).

(III). By 2.6, it remains to consider elements  $z_i + e = \alpha m_1 + \beta m_2$ , with  $\alpha + \beta = 3$ . In this case,  $C_3 \supseteq \{x + e, z_1 + e, \} \cup \{y_1, y_2, \text{ induced by } y + e\}$ .  $\diamond$

### 3 Structure of $C_3$ .

**Theorem 3.1** Assume  $|Ap_2| = 3$ . Then

1. If  $|C_3| \geq 2$ , then  $|\text{Supp}(x)| \leq 2$ , for each  $x \in C_3$ .

2. If  $|C_3| \geq 4$ , then

(a) There exist  $x_1, x_2 \in C_3$  such that  $|\text{Supp}(x_i)| = 2$ .

(b) There exist  $n_i, n_j \in Ap_1$  such that  $C_2 (= Ap_2) = \begin{cases} 2n_i \\ n_i + n_j, \\ 2n_j \end{cases}$ ,  $C_3 = \begin{cases} 3n_i \\ 2n_i + n_j \\ n_i + 2n_j \\ 3n_j \end{cases}$ .

Proof. 1. If  $x \in C_3$ , then  $|\text{Supp}(x)| \leq 3$ . First we show that if  $x_1, x_2 \in C_3$ , then  $|\text{Supp}(x_i)| \leq 2$  for  $i = 1, 2$ . Assume that  $x_1 = n_i + n_j + n_k$ ,  $n_i < n_j < n_k$ . Then, by (1.4.1a) and the assumption, we deduce that

$$Ap_2 = \{n_i + n_j, n_i + n_k, n_j + n_k\} \quad (*)$$

Hence  $2n_i, 2n_k \notin Ap$ ,  $2n_j \in Ap \iff n_i + n_k = 2n_j$ . If  $x_2 \in C_3, x_2 \neq x_1$  there are three cases:

(a) If  $|\text{Supp}(x_2)| = 1$ , hence  $x_2 = 3n_t$ , then  $2n_t \in C_2$ , by (1.4.1a). Hence by (\*)  $2n_t = \begin{bmatrix} n_i + n_j \\ n_i + n_k \\ n_j + n_k \end{bmatrix}$

In every case we see that  $|\text{Supp}(x_2)| \geq 2$ , a contradiction.

(b) If  $|\text{Supp}(x_2)| = 2$ ,  $x_2 = 2n_t + n_v, t \neq v$ , then  $2n_t \in C_2$ , by (1.4.1a).

$$\text{Hence by } (*) \ 2n_t = \begin{cases} n_i + n_j & \implies x_2 = n_i + n_j + n_v \implies n_v \in \{n_i, n_j\} \\ & n_v = n_i \implies x_2 = 2n_i + n_j \implies 2n_i \in Ap_2 \\ & n_v = n_j \implies x_2 = n_i + 2n_j \implies n_i + n_k = 2n_j \in Ap_2 \implies 2n_i \in Ap_2 \\ n_i + n_k & \text{(similar)} \\ n_j + n_k & \text{(similar)} \end{cases}$$

every case contradicts equality (\*).

(c) If  $|\text{Supp}(x_2)| = 3$ ,  $x_2 = n_t + n_v + n_w$ ,  $n_t < n_v < n_w$ , by (\*) we must have:  $n_t + n_v = n_i + n_j$ , hence  $x_2 = n_i + n_j + n_w$ . This equality implies that

$$\begin{cases} n_w \neq n_k & \text{(since } x_2 \neq x_1) \\ n_w \neq n_j & \text{(otherwise } x_2 = n_i + 2n_j = 2n_i + n_k, \text{ see } (*)) \\ n_i + n_w \in Ap_2 & \implies n_i + n_w = n_j + n_k \end{cases}$$

Hence  $x_2 = 2n_j + n_k = n_i + 2n_k$ , contradiction.

2(a). Let  $x_i \in C_3, 1 \leq i \leq 4$ , distinct elements such that  $1 \leq |\text{Supp}(x_i)| \leq 2$  (by 1). If  $|\text{Supp}(x_i)| = 1$ , for

$i = 1, 2, 3$ ,  $\begin{cases} x_1 = 3n_a \\ x_2 = 3n_i \\ x_3 = 3n_c \end{cases}$ , then by (1.4.1a)  $C_2 = Ap_2 = \{2n_a, 2n_i, 2n_c\}$ . Since  $|Ap_2| = 3$ , by (I) and (1.4.1a), we

have  $|\text{Supp}(x_4)| = 2$ : we can assume  $x_4 = 2n_a + n_d, n_a + n_d = 2n_i$ . Hence  $x_2 = n_a + n_d + n_i$ , with  $n_a \neq n_d$  and so  $|\text{Supp}(x_2)| \geq 2$ , against the assumption.

2(b). Let  $\{x_1, x_2, x_3, x_4\} \subseteq C_3$ ; by 2(a), we can assume that  $\begin{cases} x_1 = 2n_i + n_j, & n_i \neq n_j \\ x_2 = 2n_h + n_k & n_h \neq n_k \end{cases}$ .

We prove that  $|Ap_2| = 3 \implies n_i = n_k, n_j = n_h$ , i.e.  $\text{Supp}(x_1) = \text{Supp}(x_2), x_2 = n_i + 2n_j$ .

Note that  $|Ap_2| = 3 \implies |\{n_i, n_j, n_h, n_k\}| \leq 3$ , otherwise,  $\{2n_i, 2n_h, n_i + n_j, n_h + n_k\} \subseteq Ap_2 \implies |Ap_2| \geq 4$ . In fact if  $2n_i = n_h + n_k \implies |\text{Supp}(x_1 + e)| \geq 3$  (the other cases are similar). Hence there are three possible distinct

situations:

$$(b_1) \left\{ \begin{array}{l} x_1 = 2n_i + n_j, \quad n_i \neq n_j, \\ x_2 = 2n_h + n_j \quad n_h \neq n_j \end{array} \right. , \quad (b_2) \left\{ \begin{array}{l} x_1 = 2n_i + n_j, \quad n_j \neq n_i \\ x_2 = 2n_h + n_i \quad n_h \neq n_i \end{array} \right. , \quad (b_3) \left\{ \begin{array}{l} x_1 = 2n_i + n_j, \quad n_i \neq n_j \\ x_2 = 2n_i + n_k \quad n_k \neq n_j \end{array} \right. .$$

(b<sub>1</sub>). We have  $Ap_2 \supseteq \{2n_i, n_i + n_j, 2n_h, n_h + n_j\}$ , hence  $|Ap_2| = 3 \implies$   

$$\left[ \begin{array}{ll} \text{either} & n_i + n_j = 2n_h \implies x_2 = n_i + 2n_j \quad (\text{thesis}) \\ \text{or} & 2n_i = n_h + n_j \quad (\text{analogous}). \end{array} \right.$$

(b<sub>2</sub>). We have  $Ap_2 \supseteq \{2n_i, n_i + n_j, n_h + n_i, 2n_h\} \implies 2n_h = n_i + n_j \implies$  either  $x_2 = x_1$

(against the assumption), or  $n_h = n_j (\implies x_2 = 2n_j + n_i)$ .

(b<sub>3</sub>). This case cannot happen. In fact we have  $Ap_2 = \{2n_i, n_i + n_j, n_i + n_k\}$ . By similar arguments as above one can see that for any other element  $x \in C_3$ ,  $x \neq x_1, x_2$ , the maximal representation of  $x$  must be written as  $x = 3n_i$ . In fact the other possible representations are incompatible; for instance,  $x = 3n_j$ , with  $2n_j = n_k + n_i \implies x = n_i + n_j + n_k$ , impossible by (2). This would mean that  $|C_3| \leq 3$ , against the assumption.

According to the above facts we deduce that  $C_3 = \{3n_i, 2n_i + n_j, n_i + 2n_j, 3n_j\}$ .  $\diamond$

**Example 3.2** According to the notation of (3.1), we show several examples of semigroups which verify the assumptions of (3.1):  $Ap_2 = \{2n_i, n_i + n_j, 2n_j\}$ . The first example shows that the conditions (3.1.2b) are, in general, not sufficient to have  $H_R$  decreasing.

1. Let  $S = \langle 19, 21, 24, 47, 49, 50, 51, 52, 53, 54, 55, 56, 58, 60 \rangle$  and let  $n_i = 21, n_j = 24$ . Then:  $Ap_2 = \{42, 45, 48\}$ ,  $Ap_3 = \{63\}$ ,  $Ap_4 = \{84\}$ ,  $v = e - 5$ ,  $H_R = [1, 14, 14, 14, 16, 18, 19 \rightarrow]$  is non-decreasing.
2. Let  $S = \langle 19, 21, 24, 65, 68, 70, 71, 73, 74, 75, 77, 79 \rangle$  with  $n_i = 21, n_j = 24$ . Then:  
 $\ell = 2$ ,  $Ap_2 = \{42, 45, 48\}$ ,  $Ap_3 = \{3n_i, 2n_i + n_j, n_i + 2n_j, 3n_j\}$ ,  $v = e - 7$ ,  $D_2 + e = \{4n_i, 3n_i + n_j, \dots, n_i + 3n_j\}$ ,  $D_2 = \{65, 68, 71, 74, 77\}$   $H_R$  decreases at level 2,  $H_R = [1, \mathbf{12}, \mathbf{10}, 11, 15, 18, 19 \rightarrow]$ .
3. Let  $S = \langle 19, 21, 24, 46, 49, 51, 52, 54, 55, 56, 58, 60 \rangle$  with  $n_i = 21, n_j = 24$ . Then:  
 $\ell = 3$ ,  $|Ap_3| = 4$ ,  $v = e - 7$ ,  $H_R$  decreasing at level 3,  $H_R = [1, 12, \mathbf{15}, \mathbf{14}, 16, 18, 19 \rightarrow]$ .
4.  $S = \langle 30, 33, 37, 64, 68, 71, 73, 75, 76, 78, 79, 80, 82 \rightarrow 89, 91, 92, 95 \rangle$  with  $n_i = 33, n_j = 37$ . Then:  
 $Ap_2 = \{2n_i, n_i + n_j, 2n_j\}$ ,  $Ap_3 = \{3n_i, n_i + 2n_j, 3n_j\}$ ,  $Ap_4 = \{132 = 4n_i\}$ ,  $v = e - 7$ ,  $H_R$  decreases at level 4:  $H_R = [1, 23, 25, \mathbf{25}, \mathbf{24}, 27, 28, 29, 30 \rightarrow]$ .

**Lemma 3.3** Let  $d = \max\{\text{ord}(\sigma) \mid \sigma \in Ap\}$ . Assume there exists  $3 \leq r \leq d$  such that  $|Ap_r| = 1$ . Let  $r_0 := \min\{j \mid |Ap_j| = 1\}$  ( $r_0 \geq 3$ ), then:

1. If  $r_0 < d$ , there exists  $n_i \in Ap_1$  such that  $Ap_k = \{kn_i\}$  for  $r_0 \leq k \leq d$  (and  $kn_i \in Ap_k$ , for  $k < r_0$ ).
2. If there exists  $g \in D_k$ , ( $k \geq 2$ ) such that  $\text{ord}(g + e) = k + p$ , with  $p \geq r_0 - 1$ , then  $Ap_d = dn_i$  ( $n_i \in Ap_1$ ), and  $g + e = (k + p)n_i > dn_i$ ; further such element  $g$  is unique.

**Proof.** 1. By the Admissibility Theorem of Macaulay for  $H_R$ , we know that  $|Ap_k| = 1$ , for  $r_0 \leq k \leq d$ . Now suppose  $r_0 < d$ , and  $|Supp(\sigma)| \geq 2$ ,  $\sigma \in Ap_d$ . If  $n_i, n_j \in Supp(\sigma)$ , then, by (1.4), the elements  $\sigma - n_i$ ,  $\sigma - n_j$  would be distinct elements in  $Ap_{d-1}$ , contradiction.

2. By the assumption and (2.2.1 c), we have  $r_0 \leq p + 1 \leq d$  and  $|Supp(g + e)| \leq |Ap_{p+1}| = 1$ . Therefore  $g + e = (k + p)n_i$ , with  $n_i \in Ap_1$ . Then for  $r_0 \leq j \leq p + 1$ , the induced element  $jn_i \in C_j$  belongs to  $Ap_j$ , by (2.2.1 a). Then  $Ap_j = \{jn_i\}$  for  $r_0 \leq j \leq d$ . Hence  $k + p > d$ . For each  $k$ , if such  $g$  exists, then it is unique by (2.3.2).  $\diamond$

**Proposition 3.4** Assume  $|Ap_2| = 3$ ,  $|Ap_3| = 1$  and  $H_R$  decreasing. Let  $\ell = \min\{h \mid H_R \text{ decreases at level } h\}$  and let  $d = \max\{\text{ord}(\sigma) \mid \sigma \in Ap\}$ . We have:

1.  $\ell \leq d$ , there exist  $n_i, n_j \in Ap_1$ , such that  $(d+1)n_i \in D_\ell + e$ ,

$$C_2 = \begin{cases} 2n_i \\ n_i + n_j, \\ 2n_j \end{cases}, \quad C_3 = \begin{cases} 3n_i \\ 2n_i + n_j \\ n_i + 2n_j \\ 3n_j \end{cases}, \dots, \quad C_\ell = \begin{cases} \ell n_i \\ (\ell-1)n_i + n_j \\ (\ell-2)n_i + 2n_j, \\ \dots \\ \ell n_j \end{cases}$$

if  $r < \ell$ ,  $C_{r+1} = (D_r + e) \cup Ap_{r+1}$ ,

$D_\ell + e = \{ (d+1)n_i, \ell n_i + n_j, (\ell-1)n_i + 2n_j, \dots, (\ell+1)n_j \}$

and, if  $(\ell, d) \neq (3, 3)$ , then  $Ap_k = \{kn_i\}$ , for all  $k \in [3, d]$ .

2. The semigroup  $S$  is not symmetric.

Proof. 1. By (2.8.2) and (3.1.2b) we get the structure of  $C_2, C_3$ ; also,  $|C_r| \leq r+1$  for each  $r \geq 0$ , since  $Supp(C_r) \subseteq \{n_i, n_j\}$  by (1.4.1b). Further if  $H_R$  decreases at level 2, then  $|(D_2 + e)| \geq 4$ , hence there exists  $g \in D_2$  such that  $ord(g+e) \geq 4$  (since  $|C_3 \setminus Ap_3| = 3$ ): by (3.3.2) with  $r_0 = 3$ ,  $p \geq 2$ , we get  $g+e = \alpha n_i$ .

If  $H_R$  decreases at level  $\ell \geq 3$ , then for any  $3 \leq r \leq \ell$ ,  $|C_r| \geq r+1$ , by (1.5.4). Hence, for  $r \leq \ell$ :

$$C_r = \{rn_i, (r-1)n_i + n_j, \dots, rn_j\}, \quad |C_r| = r+1.$$

If  $r < \ell$  ( $\leq d$ ), then  $|D_r| \leq |C_r| = r+1$ , and  $r+1 = |C_{r+1} \setminus Ap| = |D_r^{r+1} + e|$ ; hence  $D_r^{r+1} = D_r$ . Further

$$C_{\ell+1} = \{Ap_{\ell+1}\} \cup (D_\ell^{(\ell+1)} + e). \quad (*)$$

Now, when  $\ell < d$ , there exists  $g \in D_\ell$  such that  $ord(g+e) > \ell+1$ . Hence  $g+e = \lambda n_i$  and  $Ap_d = dn_i$  by (3.3.2).

If  $\ell = d$ , then either  $D_d + e \subseteq C_{d+1}$  and  $|C_{d+1}| = d+2 \implies (d+1)n_i = g+e$  with  $g \in D_d$ ,

or there exists  $g \in D_d$ ,  $ord(g+e) > d+1$  and  $g+e = \alpha n_i$ , hence  $Ap_d = dn_i$  by (3.3.2) as above.

Now we show that  $\ell \leq d$ . If  $\ell \geq d+1$ , then  $(d+1)n_i \in C_{d+1}$ ; but  $(d+1)n_i \notin Ap \implies (d+1)n_i = d+e$ , with  $d \in D_k$ ,  $k \leq d$ , hence  $H_R$  decreases at level  $\leq d$ .

Finally we prove that  $(d+1)n_i \in D_\ell + e$ : we already know that there exists  $d \in D_\ell$  with  $d+e = \alpha n_i \notin Ap$ . Then  $\alpha \geq d+1$ , since  $\alpha n_i \in M+e$ . If  $\alpha > d+1$ , then  $ord((d+1)n_i - e) < ord(\alpha n_i - e) = \ell-1$ , i.e.  $(d+1)n_i - e \in D_k$ , with  $k < \ell$ , impossible by (\*).

2. Assume  $S$  symmetric: then it is well known that for each  $n_\alpha \in Ap$  there exist  $n_\beta \in Ap$  with  $n_\alpha + n_\beta = e + f$  ( $f$  is the Frobenius number). Clearly,  $e + f \in Ap_d$ . If  $e + f = dn_i$ , in particular there exist  $n_r, n_s \in Ap$  such that  $n_j + n_r = (n_i + n_j) + n_s = dn_i$  (because  $C_2 \subseteq Ap$ ). Hence

$$n_i + n_s = n_r \in Ap \implies \begin{cases} \text{either } n_s = n_j \implies 2n_j + n_i = dn_i \notin D_2 + e, \text{ impossible} \\ \text{or } n_s = \lambda n_i (\lambda < d-1) \implies n_j = (d-\lambda-1)n_i, \text{ impossible} \end{cases}$$

If  $(\ell, d) = (3, 3)$ , we can have  $e + f = 2n_i + n_j$ ; in this case, since  $3n_i + n_j - \mu e \in Ap_1$  (with  $\mu \in \{1, 2\}$ ), there exists  $n_r$  such that  $3n_i + n_j - \mu e + n_r = 2n_i + n_j$ , hence  $n_i + n_r = \mu e$ , impossible.  $\diamond$

**Example 3.5** According to the notation of (3.4), we show several examples of semigroups with  $v = e-4$ , or  $v = e-5$ , which verify the assumptions of (3.4):  $Ap_2 = \{2n_i, n_i + n_j, 2n_j\}$ ,  $|Ap_3| = 1$  (see Proposition 3.4.1 and next Theorem 4.10). In particular examples 2 and 3 show that, in case  $\ell = d = 3$ , we can have both  $Ap_3 = \{3n_i\}$  and  $Ap_3 \neq \{3n_i\}$

1. Let  $S = \langle 17, 19, 22, 43, 45, 46, 47, 48, 49, 50, 52, 54, 59 \rangle$  and let  $n_i = 19, n_j = 22$ . Then  $v = e-4$ ,

$$Ap_2 = \{38, 41, 44\}, Ap_3 = \{3n_i\} = \{57\}, D_2 = \{43 = 2n_i + n_j - e, 46 = n_i + 2n_j - e, 49 = 3n_j - e, 59 = 4n_i - e\}; \ell = 2, d = 3, H_R = [1, \mathbf{13}, \mathbf{12}, 13, 15, 16, 17].$$

2. Let  $S = \langle 19, 21, 24, 46, 47, 49, 50, 51, 52, 53, 54, 55, 56, 58, 60 \rangle$  and let  $n_i = 21, n_j = 24$ . Then  $v = e-4$ ,  $Ap_2 = \{42, 45, 48\}$ ,  $Ap_3 = \{63 (= 3n_i)\}$ ,  $D_2 + e = \{2n_i + n_j = 66, n_i + 2n_j = 69, 3n_j = 72\}$ ,  $C_3 = (D_2 + e) \cup \{63\}$ ,  $D_3 + e = \{4n_i, 3n_i + n_j, 2n_i + 2n_j, n_i + 3n_j, 4n_j\}$ ;  $\ell = d = 3$ ,  $H_R = [1, 15, \mathbf{15}, \mathbf{14}, 16, 18, 19 \rightarrow]$ .

3. Let  $S = \langle 19, 21, 24, 44, 46, 49, 50, 51, 52, 53, 54, 55, 56, 58, 60 \rangle$  and let  $n_i = 21, n_j = 24$ . Then  $v = e-4$ ,  $Ap_2 = \{42, 45, 48\}$ ,  $Ap_3 = \{66 (= 2n_i + n_j)\} \neq \{3n_i\}$ ,  $D_2 + e = \{3n_i = 63, n_i + 2n_j = 69, 3n_j = 72\}$ ,  $C_3 = (D_2 + e) \cup \{66\}$ ;  $\ell = d = 3$ ,  $H_R = [1, 15, \mathbf{15}, \mathbf{14}, 16, 18, 19 \rightarrow]$ .

4.  $S = \langle 30, 33, 37, 73, 75 \rightarrow 89, 91, 92, 94, 95, 98, 101 \rangle$  and let  $n_i = 33, n_j = 37$ . Then:

$$Ap_2 = \{66, 70, 74\}, Ap_3 = \{99\} = \{3n_i\}, Ap_4 = \{132\} = \{4n_i\}, v = e-5. \text{ One can check that the subsets } C_i, D_i, \text{ for } i \leq 4 \text{ have the structure described in (3.4); } \ell = d = 4, H_R = [1, 25, 25, \mathbf{25}, \mathbf{24}, 25, 27, 29, 30 \rightarrow].$$

**Theorem 3.6** Assume the Hilbert function  $H_R$  decreasing and  $|Ap_2| = 4$ . Then one of the following cases holds ( $n_i, n_j, n_k, n_h$  denote distinct elements in  $Ap_1$ ) :

- (a)  $Ap_2 = \{2n_i, n_i + n_j, n_i + n_k, n_j + n_k\}$ ,  $C_3 = \{n_i + n_j + n_k, 3n_i, 2n_i + n_j, 2n_i + n_k\}$ .
- (b)  $Ap_2 = \{2n_i, n_i + n_j, 2n_j, n_i + n_k\}$ ,  $C_3 \subseteq \{3n_i, 2n_i + n_j, n_i + 2n_j, 3n_j, 2n_i + n_k\}$ .
- (c)  $Ap_2 = \{2n_i, n_i + n_j, 2n_j, n_h + n_k\}$ ,  $C_3 = \{3n_i, 2n_i + n_j, n_i + 2n_j, 3n_j\}$ .
- (d)  $Ap_2 = \{2n_i, n_i + n_j, 2n_j, 2n_k\}$ ,  $C_3 \subseteq \{3n_i, 2n_i + n_j, n_i + 2n_j, 3n_j, 3n_k\}$ .
- (e)  $Ap_2 = \{2n_i, 2n_j, n_i + n_k, n_j + n_k\}$ ,  $C_3 = \{3n_i, 2n_i + n_k, 2n_j + n_k, 3n_j\}$ .

**Proof.** Clearly  $|Supp(Ap_2)| \geq 3$ , since  $|Ap_2| = 4$ . Assume  $H_R$  decreasing. By (2.8.1-2) and by (1.4.1 a),  $|C_3| \geq 4$  and there exist at least two elements  $x_1, x_2 \in C_3$  with  $|Supp(x_i)| \geq 2$ . We proceed step-by-step. First we assume that  $|Supp(x_1)| = 3$ ,  $x_1 = n_i + n_j + n_k$ .

**Step 1.** Let  $x_2 = n_p + n_q + n_r$ . Then we get the following induced elements  $\in Ap_2$  :  $\{n_i + n_j, n_i + n_k, n_j + n_k, n_p + n_q, n_p + n_r, n_q + n_r\}$  (1.4.1 a);  $|Ap_2| = 4 \implies n_p + n_q = n_i + n_j \implies x_2 = n_i + n_j + n_r$  therefore the induced elements in  $Ap_2$  can be written as  $\{n_i + n_j, n_i + n_k, n_j + n_k, n_i + n_r, n_j + n_r\}$ . Again we deduce that either  $n_i + n_r = n_j + n_k$ , i.e.  $x_2 = 2n_j + n_k$ , or  $n_r \in \{n_i, n_j\}$ . In any case there exists a maximal representation of  $x_2$ , with  $Supp(x_2) \subseteq Supp(x_1)$ . Then we can assume that there exists at most one element  $x$  in  $C_3$  with  $|Supp(x)| = 3$ .

**Step 2.** Let  $x_2 = 2n_p + n_q$ . Then we get the following induced elements  $\in Ap_2$  :  $\{n_i + n_j, n_i + n_k, n_j + n_k, n_p + n_q, 2n_p\}$ ,  $|Ap_2| = 4 \implies n_i + n_j = n_p + n_q$ , or  $n_i + n_j = 2n_p$ , and so, by using Step 1, we obtain  $Supp(x_2) \subseteq Supp(x_1)$ .

**Step 3.** Now assume  $|Supp(x)| \leq 2$  for each  $x \in C_3$ , then  $|Supp(x_1)| = |Supp(x_2)| = 2$ . If  $Supp(x_1) \cap Supp(x_2) = \emptyset$ ,  $x_1 = 2n_i + n_j$ ,  $x_2 = 2n_h + n_k$ , then

$$Ap_2 = \{2n_i, n_i + n_j, 2n_h, n_h + n_k\},$$

where these elements are distinct, according to the assumptions. Then  $C_3 \setminus \{x_1, x_2\} \subseteq \{3n_i, 3n_h\}$ . In fact, by (I), any other possible choice  $x \in C_3$  with  $|Supp(x)| \leq 2$ ,  $x$  distinct from  $3n_i, 3n_h$  contradicts some of the assumptions, for example :

$x = 2n_i + n_k \implies n_i + n_k = 2n_h \in Ap_2$ , impossible since  $\implies x_2 = n_i + 2n_k \implies n_i \in Supp(x_1) \cap Supp(x_2)$ ,

$x = 3n_p$ ,  $p \neq i, h \implies 2n_p = n_i + n_j$ , impossible since  $\implies |Supp(x)| = 3$ . Hence

$$C_3 = \{3n_i, 2n_i + n_j, 3n_h, 2n_h + n_k\}, \quad |C_3| = 4.$$

Note that the conditions  $H_R$  decreasing at any level  $k$ ,  $|Ap_2| = 4$ ,  $|C_3| \geq 4$ , imply  $|D_k| \geq 5$  (by the assumptions and, for  $k \geq 3$ , by (1.5.4)). Since for every  $y \in D_k + e$  we have  $y \in C_h$ ,  $h \geq k + 1 \geq 3$  and so  $y$  induces elements  $\in C_3$ , we deduce that these conditions are incompatible. Hence  $|Supp(C_3)| \leq 3$ .

**Step 4.** Now we consider the situation  $|Supp(x_1)| = 3$ .

Let  $n_i, n_j, n_k$  be distinct elements in  $Ap_1$  and let  $x_1 = n_k + n_i + n_j$ ,  $x_2 = 2n_i + n_j$ . Then

$$Ap_2 \supseteq \{2n_i, n_i + n_j, n_i + n_k, n_j + n_k\}$$

(a). If the four elements above are distinct, we deduce that  $C_3 = \{n_k + n_i + n_j, 2n_i + n_j, 3n_i, 2n_i + n_k\}$ , because we must have  $C_3 \setminus \{x_1, x_2\} \subseteq \{3n_i, 2n_i + n_k\}$  (recall that the elements of  $C_3$  induce elements in  $Ap_2$ , and  $|C_3| \geq 4$ ).

(b<sub>1</sub>). Otherwise  $n_k + n_j = 2n_i$ , then  $x_2 = n_k + 2n_j$  and so  $2n_j \in Ap_2$ . This implies

$$Ap_2 = \{2n_i, n_k + n_i, n_i + n_j, 2n_j\}.$$

In fact these elements are distinct: if not,  $2n_j = n_k + n_i$  hence  $x_1 = 3n_i = 3n_j$ . We deduce that  $C_3 \setminus \{x_1, x_2\} \subseteq \{n_k + 2n_i, n_i + 2n_j, 3n_j\}$ . In fact,  $n_k + 2n_j = 2n_i + n_j = x_2$ ; further  $2n_i + n_k = 2n_k + n_j \notin C_3$ , otherwise  $2n_k \in Ap_2 \implies 2n_k = n_i + n_j$  then  $x_1 = 3n_k = 3n_i$ . Hence

$$C_3 = \{n_k + n_i + n_j = 3n_i, 2n_i + n_j, n_i + 2n_j, 3n_j\}.$$

**Step 5.** Now we assume that  $|Supp(x)| \leq 2 \forall x \in C_3$  and that  $Supp x_1 = Supp x_2 = \{n_i, n_j\}$ .

Let  $x_1 = n_i + 2n_j$ ,  $x_2 = 2n_i + n_j$ . Then  $Ap_2 \supseteq \{2n_i, n_i + n_j, 2n_j\}$ :

(b<sub>2</sub>). If  $Ap_2 = \{2n_i, n_i + n_j, 2n_j, n_i + n_k\}$ , then  $C_3 \subseteq \{3n_i, 2n_i + n_j, n_i + 2n_j, 3n_j, 2n_i + n_k\}$ .

(c). If  $Ap_2 = \{2n_i, n_i + n_j, 2n_j, n_h + n_k\}$ ,  $n_h \neq n_k$ , then  $C_3 = \{3n_i, 2n_i + n_j, n_i + 2n_j, 3n_j\}$ .

In fact  $2n_h + n_k \notin C_3$ , otherwise  $2n_h \in Ap_2 \implies 2n_h = n_i + n_j \implies 2n_h + n_k = n_i + n_j + n_k$ , then  $|Supp(2n_h + n_k)| \geq 3$ , against the assumption.

(d) If  $Ap_2 = \{2n_i, n_i + n_j, 2n_j, 2n_k\}$ , then  $C_3 \subseteq \{3n_i, 2n_i + n_j, n_i + 2n_j, 3n_j, 3n_k\}$ .

Step 6. Finally assume  $|(Supp x_1 \cap Supp x_2)| = 1$ : two possible cases.

(b<sub>3</sub>)  $x_1 = 2n_i + n_k, x_2 = n_i + 2n_j$ , then  $Ap_2 = \{2n_i, n_i + n_j, n_k + n_i, 2n_j\}$ . In fact these four elements are distinct, otherwise  $2n_j = n_i + n_k$  would imply  $x_1 = x_2$ . Then  $C_3 \subseteq \{3n_i, 2n_i + n_j, n_i + 2n_j, 3n_j, 2n_i + n_k\}$ .

Note that statement (b) summarizes cases (b<sub>1</sub>), (b<sub>2</sub>), (b<sub>3</sub>).

(e)  $x_1 = 2n_i + n_k, x_2 = 2n_j + n_k$  then  $\{2n_i, n_i + n_k, n_j + n_k, 2n_j\} = Ap_2$ . In fact these four elements are distinct, otherwise  $2n_j = n_i + n_k$  would imply  $x_1 = 2n_i + n_k, x_2 = n_i + 2n_k$  and so  $Supp(x_1) \cap Supp(x_2) = \{n_i, n_k\}$ . We conclude that  $C_3 = \{3n_i, 2n_i + n_k, 2n_j + n_k, 3n_j\}$ .  $\diamond$

**Example 3.7** We list some semigroups verifying (3.6), the second and third ones verify also (Theorem 4.9).

1. (3.6.b) Let  $S = \langle 30, 33, 37, 73, 76, 77, 79 \rightarrow 89, 91, 92, 94, 95, 98, 101, 108 \rangle$  and let  $n_i = 33, n_j = 37, n_k = 98$ . Then:  
 $Ap_2 = \{66, 70, 74, 135\} = \{2n_i, n_i + n_j, 2n_j, n_j + n_k\}, Ap_3 = \{3n_i\}, Ap_4 = \{4n_i\}, v = e - 6$ .  
 $D_2 + e = \{103 = 2n_i + n_j, 107 = n_i + 2n_j, 111 = 3n_j\} \subseteq C_3 = \{3n_i, 2n_i + n_j, n_i + 2n_j, 3n_j\}$   
 $D_3 + e = \{165 = 5n_i, 136 = 3n_i + n_j, 140 = 2n_i + 2n_j, 144 = n_i + 3n_j, 148 = 4n_j\},$   
 $C_4 = \{4n_i, 136, 140, 144, 148\},$   
 $D_4 + e = \{169 = 4n_i + n_j, 173 = 3n_i + 2n_j, 177 = 2n_i + 3n_j, 181 = n_i + 4n_j, 185 = 5n_j, 198 = 6n_i\}.$   
 $H_R = [1, 24, \mathbf{25}, \mathbf{24}, \mathbf{23}, 25, 27, 29, 30 \rightarrow]$  decreases at levels 3 and 4.
2. (3.6.b), with  $|C_3| = 5$ . Let  $S = \langle 17, 19, 22, 31, 40, 42, 43, 45, 46, 47, 49, 52, 54 \rangle$  and let  $n_i = 19, n_j = 22, n_k = 31$ . Then:  $Ap_2 = \{38, 41, 44, 50\}, Ap_3 = \emptyset, v = e - 4$ .  
 $C_3 = D_2 + e = \{57 = 3n_i, 60 = 2n_i + n_j, 63 = n_i + 2n_j, 66 = 3n_j, 69 = 2n_i + n_k\}$ . Hence  $H_R$  decreases at level 2,  $H_R = [1, \mathbf{13}, \mathbf{12}, 14, 16, 17 \rightarrow]$ .
3. (3.6.d). Let  $S = \langle 17, 22, 29, 37, 49, 64, 69, 70, 79, 82, 84, 89, 94 \rangle$  and let  $n_i = 22, n_j = 37, n_k = 29$ . Then:  
 $Ap_2 = \{44, 58, 59, 74\}, Ap_3 = \emptyset, v = e - 4$ .  $C_3 = D_2 + e = \{66, 81, 96, 111, 87\} = \{3n_i, 2n_i + n_j, n_i + 2n_j, 3n_j, 3n_k\}$ .  $H_R = [1, \mathbf{13}, \mathbf{12}, 14, 16, 17 \rightarrow]$  decreases at level 2.

## 4 Decrease of the Hilbert function: the cases $v = e - 3, v = e - 4$ .

### 4.1 Case $v = e - 3$ .

Assume the embedding dimension  $v$  and the multiplicity  $e$  satisfy  $v = e - 3$ . With notation 1.1, the Hilbert function of the ring  $R' = R/t^e R$  is  $H_{R'} = [1, v - 1, a, b, c]$  with  $a + b + c = 3$ ; by Macaulay's theorem, the admissible cases are

- |       |         |         |         |                  |
|-------|---------|---------|---------|------------------|
| (i)   | $a = 1$ | $b = 1$ | $c = 1$ | (stretched case) |
| (ii)  | $a = 2$ | $b = 1$ | $c = 0$ |                  |
| (iii) | $a = 3$ | $b = 0$ | $c = 0$ | (short case)     |

**Remark 4.1** In cases (i), (ii) of the above table we have  $|Ap_2| \leq 2$ , hence  $H_R$  is non-decreasing by (1.5.5 c). Then the possible decreasing examples have  $H_{R'} = [1, v - 1, 3]$  (short case). In any case, it is clear that no  $R = k[[S]]$  with  $S$  symmetric and  $v = e - 3$  has decreasing Hilbert function. In fact, recall that  $S$  symmetric implies  $B = \{e + f\} = Ap_m$ .

**Theorem 4.2** With Notation 1.2, assume  $v = e - 3$ . Then the following conditions are equivalent:

1.  $H_R$  decreases.
2.  $H_R$  decreases at level 2.
3. (a) The Hilbert function of  $R' = R/t^e R$  is  $[1, e - 4, 3]$ .  
 (b) There exist distinct elements  $n_i, n_j \in Ap_1$  such that:  
 $C_2 = \{2n_i, n_i + n_j, 2n_j\} \quad D_2 + e = C_3 = \{3n_i, 2n_i + n_j, n_i + 2n_j, 3n_j\}.$

Proof. If  $H_R$  decreases at some level, then by (4.1) we have  $H_{R'} = [1, e - 4, 3]$ , hence  $Ap_3 = \emptyset$ , and so  $D_2 + e = C_3$ , by (2.2.1 b).

1  $\implies$  2. If  $H_R$  decreases at level  $j \geq 3$ , then by [5, Corollary 4.2],  $|C_h| \geq h + 1$  for each  $2 \leq h \leq j$ . In particular we get  $|C_3| \geq 4$ , hence  $|D_2| \geq 4 > |C_2| = 3$  and  $H_R$  decreases at level 2.

2  $\implies$  3. If  $H_R$  decreases at level 2, then by [5, Corollary 2.4]  $|D_2| \geq 4$ , hence  $|C_3| \geq 4$  and by applying Theorem 3.1 we deduce  $D_2 + e = C_3 = \{3n_i, 2n_i + n_j, n_i + 2n_j, 3n_j\}$ . Hence the thesis.

3  $\implies$  1. It is clear.  $\diamond$

Note that the element of  $D_2$  are an arithmetic sequence ( $x_{k+1} = x_k + n_j - n_i$ ),  $k = 1, 2, 3$ .

Under the assumption  $v = e - 3$ , Theorem 4.2 allows to prove that  $e = 13$  is the smallest multiplicity for a semigroup with decreasing Hilbert function. This bound is sharp, as shown in Examples (1.6) and (4.7).

**Proposition 4.3** *If  $v = e - 3$  and the equivalent conditions of (4.2) hold, then:*

1. *There exist ten distinct elements in  $S$  such that*

$$M \setminus 2M \supseteq \{e, n_i, n_j, 3n_i - e, 2n_i + n_j - e, n_i + 2n_j - e, 3n_j - e\}, \quad Ap_2 = \{2n_i, n_i + n_j, 2n_j\}.$$

2. *If either  $e$  is odd, or  $e=12$ , there exists  $h \geq 2$  such that  $(2n_i + 2n_j - he) \in Ap_1$ .*

3.  $e = e(S) \geq 13$ .

Proof. 1. For simplicity we denote  $n_i < n_j$  the elements  $\in S$  such that  $C_2 = \{2n_i, n_i + n_j, 2n_j\}$ . By the assumption,  $|D_2| = 4$ , and  $D_2 + e = \{3n_i, 2n_i + n_j, n_i + 2n_j, 3n_j\}$ . Hence

$$M \setminus 2M \supseteq \{e, n_i, n_j, 3n_i - e, 2n_i + n_j - e, n_i + 2n_j - e, 3n_j - e\}; \quad Ap_2 = \{2n_i, n_i + n_j, 2n_j\} (= C_2).$$

Clearly, to see that the above elements of  $M \setminus 2M$  are all distinct, it's enough to verify that  $3n_i - e \neq n_j$ : this implies  $|M \setminus 2M| \geq 7$ ,  $e \geq 10$ .

Assume  $3n_i - e = n_j$ , then  $C_2 = \{2n_i, 4n_i - e, 6n_i - 2e\}$ ,  $D_2 = \{3n_i - e, 5n_i - 2e, 7n_i - 3e, 9n_i - 4e\}$ ; hence  $M \setminus 2M \supseteq \{e, n_i, 3n_i - e, 5n_i - 2e, 7n_i - 3e, 9n_i - 4e\}$ ,  $Ap_2 = \{2n_i, 4n_i - e, 6n_i - 2e\}$ . Since  $8n_i - 3e = (3n_i - e) + (5n_i - 2e) \notin Ap_2$ , one has  $8n_i - 4e \in M$ , impossible (otherwise  $9n_i - 4e \in 2M \cap (M \setminus 2M)$ ).

2. Note that  $n_i + (n_i + 2n_j - e) = 2n_i + 2n_j - e \in 2M \setminus Ap_2$ , then  $2n_i + 2n_j - 2e \in S$ . It is easy to see that  $2n_i + 2n_j - 2e \notin \langle n_i, n_j, 3n_i - e, 2n_i + n_j - e, n_i + 2n_j - e, 3n_j - e \rangle$ .

Moreover for all  $y \in Ap_2$  and for all  $k \geq 0$ , we cannot have  $2n_i + 2n_j - 2e = y + ke$ . Hence  $2n_i + 2n_j - 2e \in (Ap_1 + he) \cup \langle e \rangle$ .

Now we show that  $2n_i + 2n_j = \lambda e$  is impossible for  $e$  odd and for  $e = 12$ .

Clearly  $e$  odd  $\implies \lambda$  even, and so  $n_i + n_j \equiv e$ , impossible since  $n_i + n_j \in Ap_2$ .

Let  $e = 12$ . Then  $n_i + n_j = 6\lambda$ ,  $\lambda$  odd (otherwise  $n_i + n_j = \mu e$ . Then  $n_j \equiv -n_i - 6 \equiv 6 + 11n_i \pmod{12}$ ). Clearly we cannot have

$\overline{n_i}, \overline{n_j} \in \{\overline{3}, \overline{4}, \overline{6}, \overline{8}, \overline{9}\}$ , hence  $\overline{(n_i, n_j)} \in \{(\overline{1}, \overline{5}), (\overline{5}, \overline{1}), (\overline{7}, \overline{11}), (\overline{11}, \overline{7})\}$ . These remaining cases are impossible, because they imply  $3n_i - e \equiv 3n_j - e$ . Hence there exists  $h \geq 2$  such that  $2n_i + 2n_j - he \in (M \setminus 2M)$ .

3. To prove that the equivalent conditions of (4.2) imply  $e \geq 13$ , we proceed in two steps. First note that  $|Ap| \geq 10$ , hence  $e \geq 10$ . By (2), we can assume that  $\begin{cases} M \setminus 2M \supseteq \{e, n_i, n_j, 3n_i - e, 2n_i + n_j - e, 2n_i + 2n_j - he, n_i + 2n_j - e, 3n_j - e\}, \\ Ap_2 = \{2n_i, n_i + n_j, 2n_j\}. \end{cases}$

*Step 1:*  $3n_i + n_j - 2e \in (M \setminus 2M)$ , for  $e \in \{10, 11, 12\}$ ; hence  $|Apéry| \geq 12$  and we cannot have  $e \in \{10, 11\}$ .  $3n_i + n_j - e = (3n_i - e) + n_j \notin Ap_2$ : (otherwise  $n_j = 3n_i - e$ ). Hence  $3n_i + n_j - 2e \in S$ . It is easy to check that  $3n_i + n_j - 2e \notin \langle n_i, n_j, 3n_i - e, 2n_i + n_j - e, n_i + 2n_j - e, 3n_j - e, 2n_i + 2n_j - 2e \rangle \cup (Ap_2 + ke)$ ,  $k \geq 0$ . If  $3n_i + n_j = \lambda e$ ,  $\lambda > 4$ , then  $\overline{3n_j} = \overline{3n_j - e} = \overline{-9n_i} \pmod{e}$ . This is impossible for  $e \in \{10, 11, 12\}$ :

$e = 10 \implies \overline{-9n_i} = \overline{n_i}$ , impossible since  $n_i, 3n_j - e \in (M \setminus 2M)$ .

$e = 11 \implies \overline{-9n_i} = \overline{2n_i}$ , impossible since  $2n_i, 3n_j - e \in Ap$ .

$e = 12 \implies \overline{-9n_i} = \overline{3n_i}$ , impossible since  $3n_i - e, 3n_j - e \in (M \setminus 2M)$ .

Hence there exists  $k \geq 2$  such that  $3n_i + n_j - ke \in (M \setminus 2M)$ . Hence  $v \geq 8$  for  $e \in \{10, 11, 12\}$ .

*Step 2.* Now assume  $e = 12$ ; we prove that there exists  $q \geq 3$  such that  $3n_i + 2n_j - qe \in (M \setminus 2M)$ , hence  $|Apéry| \geq 13$ , therefore we cannot have  $e = 12$ . First, we know, by (2) and step 1, that there exist  $h, k \in \mathbb{N}$  such that



$(M \setminus 2M) \supseteq \{e, n_i, n_j, 3n_i - e, 2n_i + n_j - e, n_i + 2n_j - e, 2n_i + 2n_j - he, 3n_i + n_j - ke, 3n_j - e\}$ .

Let  $r := \max\{h, k\}$ : one can see that the element  $3n_i + 2n_j - re \in 2M \setminus (Ap_2 + \mathbb{N}e)$  and that  $3n_i + 2n_j - (r+1)e \notin \{e, n_i, n_j, 3n_i - e, 2n_i + n_j - e, n_i + 2n_j - e, 2n_i + 2n_j - he, 3n_i + n_j - ke, 3n_j - e\}$ .

Hence it belongs to  $(Ap_1 + \mathbb{N}e) \cup \mathbb{N}e$ .

Finally it results that  $3n_i + 2n_j \notin \mathbb{N}e$ . Otherwise,  $\overline{3n_i - e} = \overline{-2n_j} = \overline{10n_j} \implies 10n_j - 3n_i = 12k \implies 10n_j = 3(n_i + 4k) \implies n_j = 3h \implies h$  odd. But this implies that  $\overline{3n_i - e} = \overline{2n_j}$ , impossible. Hence there exists  $q \geq r + 1 \geq 3$  such that  $3n_i + 2n_j - qe \in (M \setminus 2M)$ . This proves that  $e \geq 13$ .  $\diamond$

**Corollary 4.4** *If  $v = e - 3$  and either  $|Ap_2| \leq 2$ , or  $(|Ap_2| = 3, e \leq 12)$  the Hilbert function is non-decreasing.*

In the next example, for  $v = e - 3$ , we exhibit a technique of computation which allows at first to verify that  $H_R$  decreasing  $\implies e \geq 13$  and further to give a complete description of the semigroups  $S$  with  $e = 13 = v + 3$  and decreasing Hilbert function. Similar tables could be used for  $e > 13$  and also if  $v = e - r, r \geq 4$  to find semigroups with  $H_R$  decreasing.

**Example 4.5** Let  $v = e - 3$  and  $H_R$  decreasing;  $e \geq 10$  (4.3) and there exist  $n_i, n_j \in Ap_1$ , distinct elements such that  $Ap_1 \supseteq \{e, n_i, n_j, 3n_i - e, 2n_i + n_j - e, n_i + 2n_j - e, 3n_j - e\}$ ,  $Ap_2 = \{2n_i, n_i + n_j, 2n_j\}$ .

Let  $GCD(e, n_i) = 1$ ; the following table is useful to find the pairs  $(n_i, n_j)$  "compatible" with the assumption on the Apéry set. In the table we fix  $n_j \equiv h n_i \pmod{e}$ : in the columns we indicate the classes of elements of the Apéry set  $\pmod{n_i}$ : in each row we must have distinct values  $\pmod{e}$ .

Under our conditions, we consider  $4 \leq h \leq e - 1$  and  $10 \leq e \leq 13$ . Clearly, for  $e > 10$  some element in  $\{2n_i + 2n_j - \lambda_1 e, 3n_i + n_j - \lambda_2 e, n_i + 3n_j - \lambda_3 e, 3n_i + 2n_j - \lambda_4 e, 2n_i + 3n_j - \lambda_5 e\}$  must belong to  $Ap_1$ , for this reason we add 5 columns useful to complete the Apéry set in cases  $11 \leq e \leq 13$ .

$n_i$	$2n_i$	$3n_i$	$n_j$	$n_i + n_j$	$2n_i + n_j$	$2n_j$	$n_i + 2n_j$	$3n_j$	$2n_i + 2n_j$	$3n_i + n_j$	$n_i + 3n_j$	$3n_i + 2n_j$	$2n_i + 3n_j$	
1	2	3	$h$	$(h+1)$	$(h+2)$	$2h$	$(2h+1)$	$3h$	$(2h+2)$	$(h+3)$	$(3h+1)$	$(2h+3)$	$(3h+2)$	
1	2	3	4	5	6	8	9	12	10	7	13	11	14	ok
1	<b>2</b>	3	5	6	7	10	11	<b>15</b>	12	8	16			no
1	2	3	6	7	8	12	<b>13</b>	18	14	9	19			no
1	<b>2</b>	3	7	8	9	14	<b>15</b>	21	16	10	22			no
1	2	<b>3</b>	8	9	10	<b>16</b>	17	24	18	11	25			no
1	2	3	9	10	11	18	19	<b>27</b>	20	12	28			no
1	2	3	10	11	12	20	21	30	22	13	5	23	32	ok
1	2	3	11	12	<b>13</b>	22	23	33	24	14	34			no
1	2	3	12	<b>13</b>	14	24	25	36	26	15	37			no

The table shows that for each  $e$ , only few cases with  $H_R$  decreasing are "admissible". Moreover with some other check in cases  $e \in \{10, 11, 12\}$ , one can confirm that  $e \geq 13$ , as proved in (4.3.3);

in case  $e = 10$  the only remaining case is  $(n_i = 2, n_j = 5)$ , impossible since this would imply  $2n_j \equiv 0$ ;

in case  $e = 12$  the cases to be verified are  $(n_i, n_j) \in \{(2, n_j = 3)(2, 9), (4, -), (6, -), (8, -)\}$ , which are clearly incompatible with the assumptions (for  $(2, 9) : 2n_j \equiv 3n_i$ ).

For  $e = 13$ , the possible cases are  $n_j \equiv 4n_i$ , or  $n_j \equiv 10n_i$  (in the table the pairs of distinct elements with the same class  $\pmod{n_i}$  for  $e = 13$  are written in bold).

By means of these computations we deduce the structure of such semigroup rings with  $H_R$  decreasing:

**Proposition 4.6** *With Notation 1.2, let  $R = k[[S]]$ , with multiplicity  $e = 13$  and  $v = 10 (= e - 3)$ . Further let  $1 \leq p \leq 12$ : there exists  $R_p$  satisfying the equivalent conditions of Theorem 4.3 if and only if there exist  $(k, k', \alpha, \beta, \gamma)_p$ ,  $k, \alpha, \beta, \gamma \in \mathbb{N}$ ,  $k' \in \mathbb{Z}$ , such that semigroup  $S_p$  has the following minimal set of generators.*

$$S_p = \langle e, n_i, n_j, 3n_i - e, 2n_i + n_j - e, n_i + 2n_j - e, 3n_j - e, 2n_i + 2n_j - \alpha e, 3n_i + n_j - \beta e, 3n_i + 2n_j - \gamma e \rangle$$

$$\text{with } n_i = ke + p, k \geq 1, n_j = k'e + 4p, -2 \leq k' \leq 4k - 2, 4k' > 3k - p, \alpha < \gamma, \beta < \gamma.$$

Proof. By (4.5), for  $e = 13$ , the possible cases are  $n_j \equiv 4n_i$ , or  $n_j \equiv 10n_i$ : these conditions are symmetric,  $n_i \equiv 4n_j \iff n_j \equiv 10n_i$ , hence there is only one class of semigroups with decreasing Hilbert function. Further :

-  $n_j = k'e + 4p = 4n_i - re \implies r \geq 2$  (otherwise  $r = 1 \implies n_j = (3n_i - e) + n_i \in 2M$ ).

-  $3n_i \equiv 4n_j \implies 3n_i - e < n_j + (3n_j - e) \implies 3n_i < 4n_j$ , i.e.,  $3k < 4k' + p$ .

The remaining 3 generators must be equivalent to  $2n_i + 2n_j, 3n_i + n_j, 3n_i + 2n_j \pmod{e}$ . Let  $2n_i + 2n_j - \alpha e \in Ap_1$ , then  $\alpha \geq 2$  ( $\alpha = 1 \implies 2n_i + 2n_j - e = n_j + (2n_i + n_j - e) \notin Ap_1$ ); now note that  $3n_i + 2n_j - \gamma e = (2n_i + 2n_j - \alpha e) + n_i + (\alpha - \gamma)e \notin M + e \implies \gamma > \alpha$ ; analogously,  $3n_i + 2n_j - \beta e = (3n_i + n_j - \beta e) + n_j + (\beta - \gamma) \implies \gamma > \beta$ .  
 $\diamond$

We exhibit, in case  $e = 13 = v + 3$ , for each  $1 \leq p \leq 12$  an example of semigroup  $S_p$  as in (4.6).

**Example 4.7** With the notation of (4.6) above, let  $k = 1$ ,  $1 \leq p \leq 12$ ; the following semigroups  $S_p$  with  $k'$  minimal ( $-2 \leq k' \leq 1$ ) have decreasing Hilbert function ( $S_6$  is Example 1.6).

$(\bar{n}_i, \bar{n}_j) = (\bar{1}, \bar{4})$	: $S_1 = \langle 13, 14, 17, 29, 32, 33, 35, 36, 37, 38 \rangle$	$k' = 1$	$c = 26$
$(\bar{n}_i, \bar{n}_j) = (\bar{2}, \bar{8})$	: $S_2 = \langle 13, 15, 21, 32, 38, 40, 44, 46, 48, 50 \rangle$	$k' = 1$	$c = 38$
$(\bar{n}_i, \bar{n}_j) = (\bar{3}, \bar{12})$	: $S_3 = \langle 13, 16, 25, 35, 44, 47, 53, 56, 59, 62 \rangle$	$k' = 1$	$c = 50$
$(\bar{n}_i, \bar{n}_j) = (\bar{4}, \bar{3})$	: $S_4 = \langle 13, 17, 16, 35, 36, 37, 38, 40, 41, 44 \rangle$	$k' = 0$	$c = 32$
$(\bar{n}_i, \bar{n}_j) = (\bar{5}, \bar{7})$	: $S_5 = \langle 13, 18, 20, 41, 43, 45, 47, 48, 50, 55 \rangle$	$k' = 0$	$c = 43$
$(\bar{n}_i, \bar{n}_j) = (\bar{6}, \bar{11})$	: $S_6 = \langle 13, 19, 24, 44, 49, 54, 55, 59, 60, 66 \rangle$	$k' = 0$	$c = 54$
$(\bar{n}_i, \bar{n}_j) = (\bar{7}, \bar{2})$	: $S_7 = \langle 13, 20, 28, 47, 55, 62, 63, 70, 71, 77 \rangle$	$k' = 0$	$c = 65$
$(\bar{n}_i, \bar{n}_j) = (\bar{8}, \bar{6})$	: $S_8 = \langle 13, 21, 19, 44, 46, 48, 50, 54, 56, 62 \rangle$	$k' = -1$	$c = 50$
$(\bar{n}_i, \bar{n}_j) = (\bar{9}, \bar{10})$	: $S_9 = \langle 13, 22, 23, 53, 54, 55, 56, 63, 64, 73 \rangle$	$k' = -1$	$c = 61$
$(\bar{n}_i, \bar{n}_j) = (\bar{10}, \bar{1})$	: $S_{10} = \langle 13, 23, 27, 56, 60, 64, 68, 70, 74, 84 \rangle$	$k' = -1$	$c = 72$
$(\bar{n}_i, \bar{n}_j) = (\bar{11}, \bar{5})$	: $S_{11} = \langle 13, 24, 31, 59, 66, 73, 77, 80, 84, 95 \rangle$	$k' = -1$	$c = 83$
$(\bar{n}_i, \bar{n}_j) = (\bar{12}, \bar{9})$	: $S_{12} = \langle 13, 25, 22, 53, 56, 59, 62, 68, 71, 80 \rangle$	$k' = -2$	$c = 68$

## 4.2 Case $v = e - 4$ .

With notation 1.1 and 1.2, the Hilbert function of  $R' = R/t^e R$ , is  $H_{R'}(z) = [1, v-1, a, b, c, d]$  with  $a+b+c+d = 4$ .

**4.8** By Macaulay's Theorem the admissible  $H_{R'}$ , are

$$\begin{aligned} & [1, v-1, 4] \\ & [1, v-1, 3, 1] \\ & [1, v-1, 2, 2] \\ & [1, v-1, 2, 1, 1] \\ & [1, v-1, 1, 1, 1, 1] \quad (\text{stretched}) \end{aligned}$$

When  $|Ap_2| \leq 2$  we know by (1.5.5 c) that  $H_R$  is non-decreasing. Hence we consider the first two cases.

**Theorem 4.9** With Notation 1.2, assume  $v = e - 4$ ,  $|Ap_2| = 4$ ,  $Ap_3 = \emptyset$ . The following conditions are equivalent:

1.  $H_R$  decreases at level 2.
2. There exist  $n_i, n_j, n_k \in Ap_1$ , distinct elements, such that

$$\text{either } Ap_2 = \{2n_i, n_i + n_j, 2n_j, n_i + n_k\}, \quad C_3 = D_2 + e = \{3n_i, 2n_i + n_j, n_i + 2n_j, 3n_j, 2n_i + n_k\}$$

$$\text{or } Ap_2 = \{2n_i, n_i + n_k, 2n_j, 2n_k\}, \quad C_3 = D_2 + e = \{3n_i, 2n_i + n_j, n_i + 2n_j, 3n_j, 3n_k\}$$

Proof. By (2.2.1 b) we have  $C_3 = D_2 + e$ , then  $H_R$  decreases at level 2  $\iff |D_2| \geq 5$ , i.e.  $|C_3| \geq 5$ ; now apply (3.6).  $\diamond$

**Theorem 4.10** With Notation 1.2, assume  $v = e - 4$ ,  $|Ap_2| = 3$ ,  $|Ap_3| = 1$ . The following conditions are equivalent:

1.  $H_R$  decreases.
2.  $H_R$  decreases at level  $h \leq 3$ .

3. There exist  $n_i, n_j \in Ap_1$ , distinct elements, such that

$$Ap_2 = \begin{cases} 2n_i \\ n_i + n_j \\ 2n_j \end{cases}, \quad C_3 = \begin{cases} 3n_i \in Ap_3 \\ 2n_i + n_j \\ n_i + 2n_j \\ 3n_j \end{cases}, \quad D_h + e = \begin{cases} 4n_i \\ hn_i + n_j \\ (h-1)n_i + 2n_j \\ \dots \\ (h+1)n_j \end{cases}, \quad (h = 2 \text{ or } h = 3).$$

Proof.  $1 \implies 2$ , and  $1 \implies 3$  follow by (3.4.4), with  $d = 3$ .

$3 \implies 2 \implies 1$  are obvious.  $\diamond$

**Corollary 4.11** *If  $R = k[[S]]$  is Gorenstein with  $v \geq e - 4$ , then  $H_R$  is non decreasing.*

Proof. For  $v \geq e - 2$ , see [6] [7]. For  $v = e - 3$  see 4.1. In case  $v = e - 4$  the result follows by (4.8), (1.5.5c), the properties of symmetric semigroups and (3.4.4b).  $\diamond$

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